Sensitivity analysis for hyperbolic equations with discontinuous solutions

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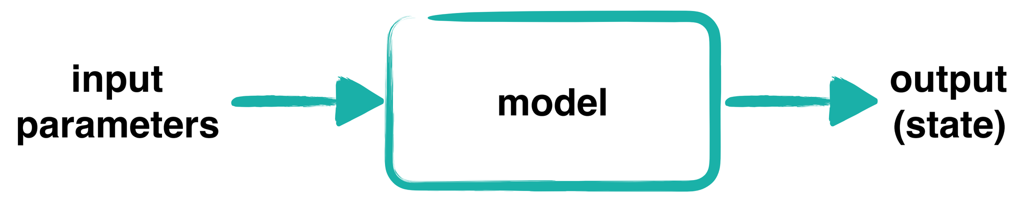
Introduction to Sensitivity Analysis

Sensitivity Analysis: study of how changes in the **inputs** of a model affect the **outputs**.



· Parameter of interest: a,
· State: $\mathbf{U}$,
· Model: system of hyperbolic equations.

Sensitivity: $\frac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$

Continuous Sensitivity Equation Method

Standard technique under hypothesis of regularity of U

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0} & \Omega \times (0,T) \\ \mathbf{U}(x,0) = \mathbf{U}_0(x) & \Omega \end{cases} \quad -\partial_a \rightarrow \quad \begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U},\mathbf{U}_a) = \mathbf{0} & \Omega \times (0,T) \\ \mathbf{U}_a(x,0) = \mathbf{U}_{a,0}(x) & \Omega \end{cases}$$

If these techniques are applied to hyperbolic equations in case of shocks, **Dirac delta functions** appear in the sensitivity. In order to have a sensitivity system which is valid also when the state is discontinuous, we add a **correction term** [2].

Sensitivity system with correction term

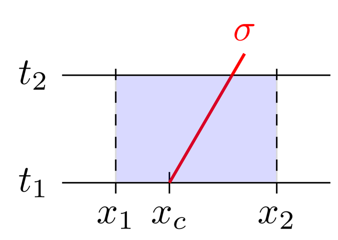
$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U},\mathbf{U}_a) = \mathbf{S} & \Omega \times (0,T) \\ \mathbf{U}_a(x,0) = \mathbf{U}_{a,0}(x) & \Omega \end{cases}$$

This system is valid in the usual sense of weak solutions also in the case of discontinuous state variables [1].

Source term definition

$$\mathbf{S} = \sum_{i=1}^{N_s} \boldsymbol{\rho}_k(t)\delta(x - x_{s,k}(t))$$

· N_s: number of discontinuities,
· $x_{k,s}(t)$: position of the k^{th} shock at time t,
· $\boldsymbol{\rho}_k$: amplitude of the k^{th} correction.



Integrating the sensitivity equations over the control volume: $\boldsymbol{\rho} = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma + \mathbf{F}_a^+ - \mathbf{F}_a^-$.
Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma = \mathbf{F}^+ - \mathbf{F}^-$.
Differentiating them with respect to the parameter one has:

$$(\mathbf{U}_a^+ - \mathbf{U}_a^-)\sigma_k + (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\nabla \mathbf{U}^+ - \nabla \mathbf{U}^-)\partial_a x_{k,s}(t) =$$
$$= \mathbf{F}_a^+ - \mathbf{F}_a^- + \left(\frac{\partial \mathbf{F}(\mathbf{U}^+)}{\partial \mathbf{U}} \nabla \mathbf{U}^+ - \frac{\partial \mathbf{F}(\mathbf{U}^-)}{\partial \mathbf{U}} \nabla \mathbf{U}^- \right) \partial_a x_{k,s}(t).$$

The terms in green are difficult to estimate due to $\partial_a x_{k,s}(t)$.
They are zero if we consider a solution $\mathbf{U}$ which is **constant** in the left and right neighbourhoods of the shock. We obtain therefore a simpler formula: $(\mathbf{U}_a^+ - \mathbf{U}_a^-)\sigma_k + (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} = \mathbf{F}_a^+ - \mathbf{F}_a^-$.
Finally, we have:

$$\boldsymbol{\rho}_k(t) = \sigma_{k,a}(\mathbf{U}^+ - \mathbf{U}^-).$$

Remark: shock detectors are necessary.

Numerical methods

Remark: the state and the sensitivity system are solved separately, because the global system is non-strictly hyperbolic.
An **HLLC-type** scheme is required for the state.

Approximate Riemann solver for the state

Roe Riemann solver: first and second order MUSCL-type implementation.

- $\lambda_1^{ROE} = \tilde{u} - \tilde{c} \quad \lambda_2^{ROE} = \tilde{u} \quad \lambda_3^{ROE} = \tilde{u} + \tilde{c}$ Roe-averaged eigenvalues.
- $\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \alpha_i \tilde{\mathbf{r}}_i$ decomposition along Roe-averaged eigenvectors.
- Star states: $\mathbf{U}_L^* = \mathbf{U}_L + \alpha_1 \tilde{\mathbf{r}}_1 \qquad \mathbf{U}_R^* = \mathbf{U}_R - \alpha_3 \tilde{\mathbf{r}}_3$.

No constraints on the approximate Riemann solver for the sensitivity.

Approximate Riemann solver for the sensitivity

HLL scheme: first and second order MUSCL-type implementation.

$$\mathbf{U}^*_{a,j-1/2} = \frac{1}{\lambda_3^{ROE} - \lambda_1^{ROE}} \left(\lambda_3^{ROE} \mathbf{U}^n_{a,j} - \lambda_1^{ROE} \mathbf{U}^n_{a,j-1} - \mathbf{F}_a(\mathbf{U}_j, \mathbf{U}_{a,j}) + \mathbf{F}_a(\mathbf{U}_{j-1}, \mathbf{U}_{a,j-1}) + \mathbf{S}_{j-1/2} \right),$$

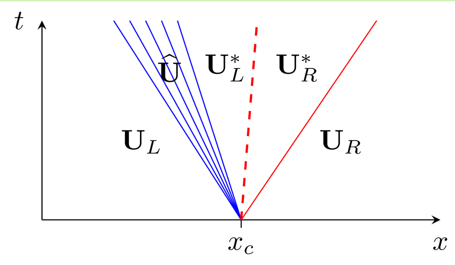
$$\begin{aligned} \mathbf{S}_{j-1/2} =& \partial_a \lambda^{ROE}_{1,j-1/2}(\mathbf{U}^*_{L,j-1/2} - \mathbf{U}_{j-1})d_{1,j-1/2} \\ &+ \partial_a \lambda^{ROE}_{2,j-1/2}(\mathbf{U}^*_{R,j-1/2} - \mathbf{U}^*_{L,j-1/2}) \\ &+ \partial_a \lambda^{ROE}_{3,j-1/2}(\mathbf{U}_j - \mathbf{U}^*_{R,j-1/2})d_{3,j-1/2}. \end{aligned}$$

Both **diffusive** and **anti-diffusive** versions of all the methods are implemented.

Convergence test case: Sod shock tube

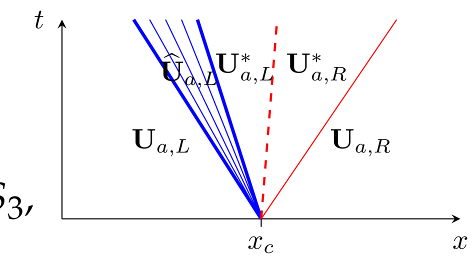
The state system:

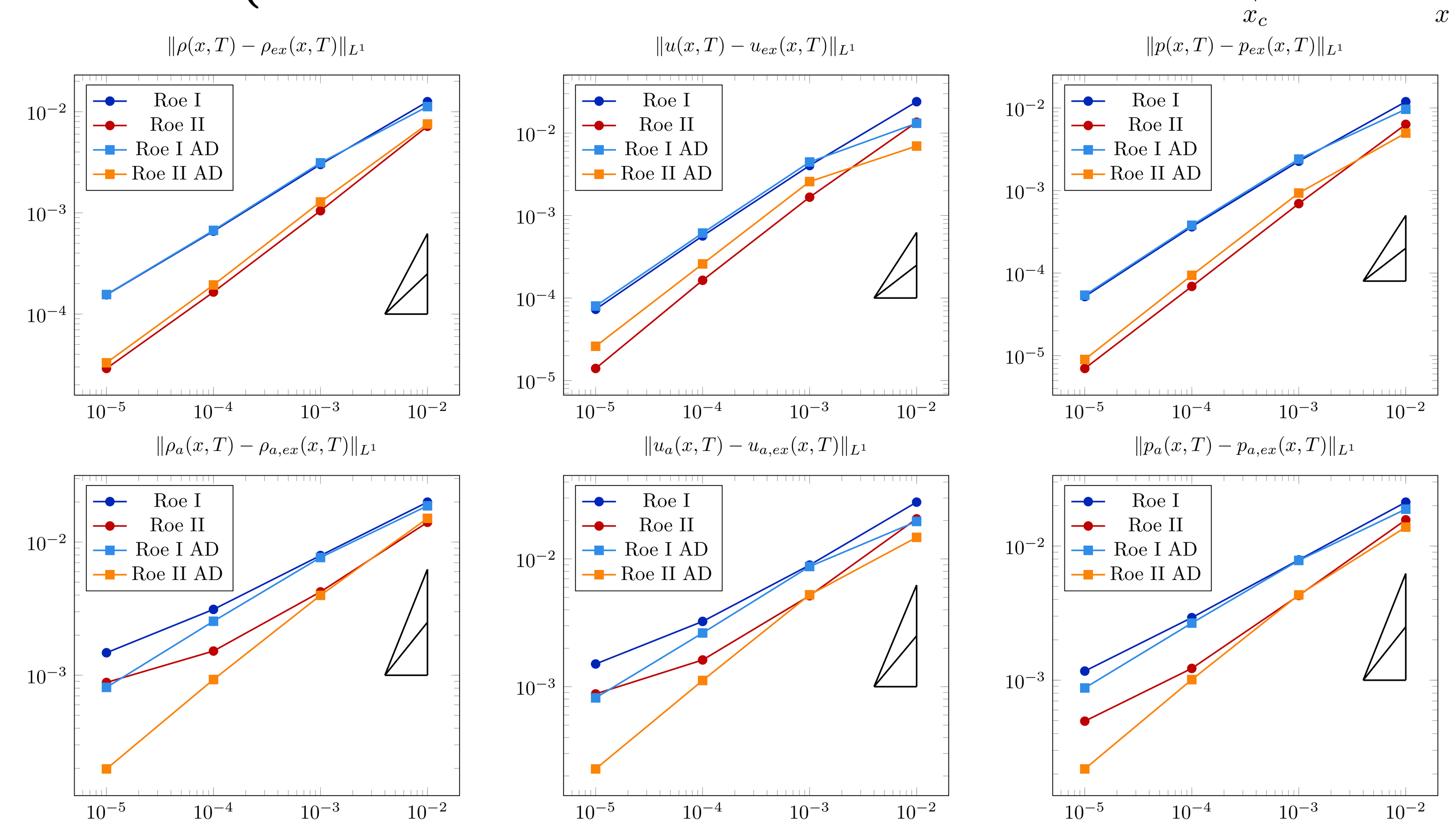
$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = 0, \end{cases}$$



The sensitivity system:

$$\begin{cases} \partial_t \rho_a + \partial_x(\rho u)_a = S_1, \\ \partial_t(\rho u)_a + \partial_x(\rho_a u^2 + 2\rho u u_a + p_a) = S_2, \\ \partial_t(\rho E)_a + \partial_x(u_a(\rho E + p) + u((\rho E)_a + p_a)) = S_3, \end{cases}$$





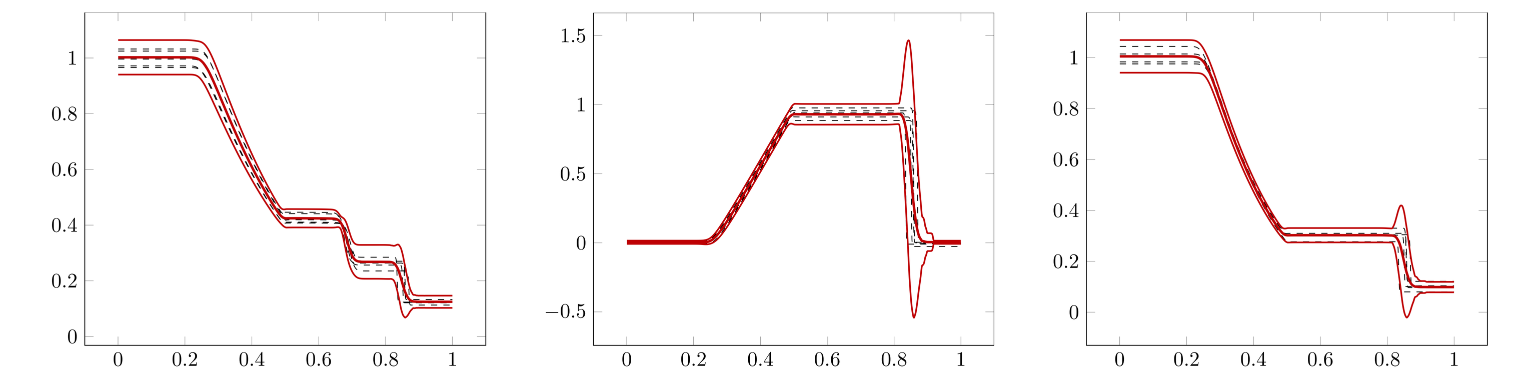
Remark: the **numerical viscosity** plays an important role in the convergence of the sensitivity.

Uncertainty quantification: Monte Carlo vs SA

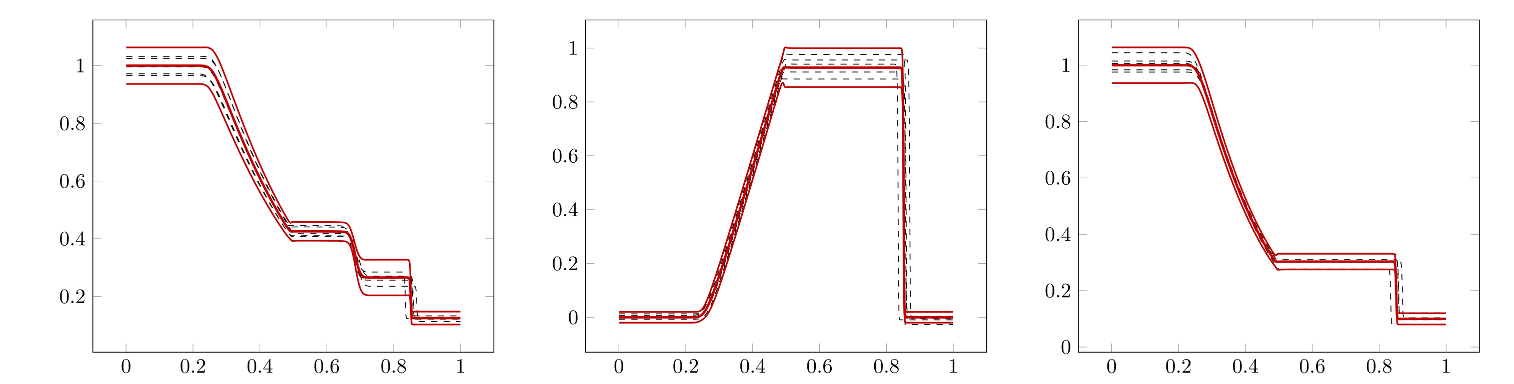
We consider a Riemann problem with uncertain parameters $\mathbf{a} = (\rho_L, \rho_R, u_L, u_R, p_R, p_L)^t$.
The aim is to determine a **confidence interval** for the variable X: $CI_X = [\mu_X - \kappa\sigma_X, \mu_X + \kappa\sigma_X]$.

MC: $\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \quad \sigma_X^2 = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$ **SA**: $\mu_X = X \quad \sigma_X^2 = \sum_{i=1}^{6} X_{a_i}^2 \sigma_{a_i}^2$

Monte Carlo method: ρ, u, and p



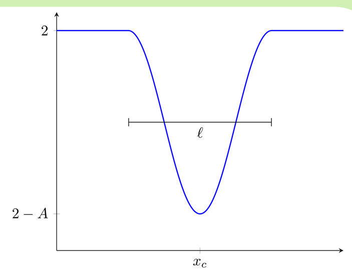
Sensitivity analysis method: ρ, u, and p



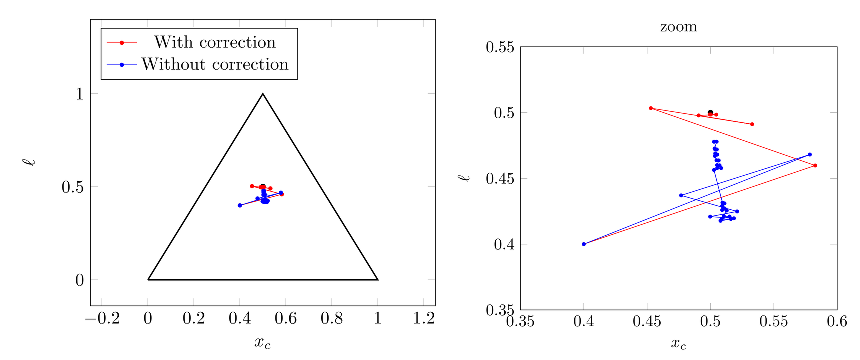
Optimization

Quasi-1D Euler system:

$$(1) \quad \begin{cases} \partial_t(h\rho) + \partial_x(h\rho u) = 0, \\ \partial_t(h\rho u) + \partial_x(h\rho u^2 + p) = p\partial_x h, \\ \partial_t(h\rho E) + \partial_x(hu(\rho E + p)) = 0, \end{cases}$$



- Cost functional $J(\mathbf{U}) = \frac{1}{2}\|p - p^*\|^2_{L^2}$
- Parameters $\mathbf{a} = (x_c, \ell)$
- Target pressure $p^* = p(x_c = 0.5, \ell = 0.5)$
- Gradient $\nabla_{\mathbf{a}} J(\mathbf{U}) = \begin{bmatrix} (p - p^*, p_{x_c})_{L^2} \\ (p - p^*, p_L)_{L^2} \end{bmatrix}$



Problem: $\min_{\mathbf{a} \in \mathcal{A}} J(\mathbf{U}(a))$ subject to (1).

Bibliography

[1] C. Bardos and O. Pironneau. A formalism for the differentiation of conservation laws. *Compte rendu de l'Académie des Sciences*, 335(10):839–845, 2002.

[2] C. Fiorini, C. Chalons, and R. Duvigneau. Sensitivity equation method for Euler equations in presence of shocks applied to uncertainty quantification. *Journal of Computational Physics*, 2018. Submitted.