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ORIGINAL ARTICLE

Optimization of Running Strategies According to the Physiological Parameters for a Two-Runners Model

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Abstract In order to describe the velocity and the anaerobic energy of two runners competing against each other for middle-distance races, we present a mathematical model relying on an optimal control problem for a system of ordinary differential equations. The model is based on energy conservation and on Newton's second law: resistive forces, propulsive forces and variations in the maximal oxygen uptake are taken into account. The interaction between the runners provides a minimum for staying 1 m behind one's competitor. We perform numerical simulations and show how a runner can win a race against someone stronger by taking advantage of staying behind, or how they can improve their personal record by running behind someone else. Our simulations show when it is the best time to overtake, depending on the difference between the athletes. Finally, we compare our numerical results with real data from the men's 1500 m finals of different competitions.

Keywords Optimization · Running strategies · Mathematics of sport · Optimal control · Middle-distance races

1 Introduction

The running strategy to win an Olympic medal is quite complex. It relies on outstanding physiology, good preparation, psychological factors and the optimal way to compete with the others to beat them. Quite a few mathematical works starting with Keller's (1974) have analysed running strategies for a single runner (Aftalion and Bonnans 2014; Behncke 1993; Mathis 1989; Morton 1986; Woodside 1991), but very few take

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into account the competition situation where the point is to beat the others (Kyle 1979; Pitcher 2009).

The point of view of Keller (1974) is to write the equations governing the energy and the velocity of a single runner, starting from Newton's second law and energy conservation. He considers a simple problem, in which the athlete runs alone on a straight path of length D, and the aim is to minimize the time T when the runner reaches the final distance D. This model matches the final times of world records. However, some hypotheses are not physiologically reasonable, and this leads to a non-realistic velocity profile. Some authors have tried to improve Keller's model: Woodside (1991) and Mathis (1989) introduce a correction by adding a fatigue term for long races. Ward-Smith (1985) and Morton (1986, 1996, 2006) follow a different approach. Morton has introduced a three component model to take into account the variations in the oxygen uptake ($\dot{V}O_2$) but the full optimal control problem is not solved. Behncke (1993) incorporates the hydraulic model of Morton to a biomechanical model that extends the ones of Keller and Ward-Smith: it is more detailed in terms of resistive forces and takes into account the reaction time of the athlete. Aftalion and Bonnans (2014) improve the models of Keller (1974), Behncke (1993) and Morton (1986, 1996, 2006) and assume that maximal oxygen uptake is a function of the anaerobic energy of the athlete. The aim is to match experimental measurements, in particular, in (Hanon et al. 2008; Hanon and Thomas 2011; Thomas et al. 2005), where Hanon et al. show how the oxygen uptake varies during races of 400, 800 and 1500 m. They solve the full numerical control problem using an optimal control solver Bocop.

As pointed out by Pitcher (2009), in middle-distance running, it is common practice to try to position oneself behind but within striking distance of the leader for most of the race and then overtake them near the finish line. Pitcher explains that the runner behind can take advantage of the slipstream of the runner in front and relies on analyses of Pugh (1971) and Kyle (1979). We believe that it is a combination of slipstream and psychological factors which explain why it is better to stay behind, and the equations can incorporate all this. The weakness of Pitcher's paper is that she imposes a strategy for one of the runners and allows only the second runner to have a free strategy. Therefore, in this paper, based on the recent work of Aftalion and Bonnans (2014), we extend the model of Pitcher (2009) to include slipstream and psychological factors in a two-runners race and we set a realistic optimal control problem with each runner having a free strategy.

1.1 Mathematical Model for a Single Runner

The system of Keller couples together the velocity of the runner at instant t, $v(t)$, the energy of the runner at instant t, $e(t)$, and the propulsive force of the runner at instant t, $f(t)$. The first equation is Newton's second law: it involves the propulsive force and the friction. Here, τ is a constant coefficient which gathers together all the friction effects, supposed to be linear in v. The friction term can be modified to include air resistance (Kyle 1979) which adds a term in $-\mathrm{cv}^2$ to the first equation.

The second equation is an energy balance incorporating the oxygen uptake, σ, considered constant in Keller's paper, whilst the second term is the work of the propulsive force f. Both equations are normalized with respect to the runner's mass:

$$\begin{cases} \dot{v}(t) = f(t) - \frac{v(t)}{\tau} & v(0) = 0, x(0) = 0,\ x(T) = D \\ \dot{e}(t) = \sigma - f(t)v(t) & e(0) = \mathrm{e}^0. \end{cases} \tag{1}$$

We use the dot notation to indicate the derivative with respect to time, i.e. $\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}$ and $\dot{e} = \frac{\mathrm{d}e}{\mathrm{d}t}$. We have set $x(t)$ to be the position so that $\dot{x} = v$. The final time T is defined as the time to reach the distance D. Moreover, e^0 is the initial energy. It is necessary to add some physiological constraints to the system (1):

- the energy must be positive:

$$e(t) \geq 0 \quad \forall t \geq 0; \tag{2}$$

- the propulsive force has an upper bound which depends on the runner's physiology, and a positive lower bound due to the fact that they are moving forwards:

$$0 \leq f(t) \leq f_\mathrm{M} \quad \forall t \geq 0. \tag{3}$$

Therefore, in this model, the athlete is identified by four parameters: e^0, the initial energy, τ, the friction coefficient, σ, the oxygen uptake, and f_M the maximal propulsive force. The aim is to solve (1)–(2)–(3) in such a way that, given a distance D, the final time T is minimal. From a mathematical point of view, it is a problem of optimal control: the propulsive force f is the control variable and the time T is the cost functional to be minimized, which depends on f through the states variables v and e. Therefore, the problem can be written as follows:

$$\min_{f \in \mathcal{F}} T(f) \quad \text{s.t.} \quad (1)\text{–}(2), \tag{4}$$

where $\mathcal{F}$ is the set of admissible controls:

$$\mathcal{F} = \{f\colon 0 \leq f(t) \leq f_\mathrm{M} \quad \forall t \geq 0\}.$$

In his work, Keller (1974) claims that his energy balance takes into account only the aerobic energy (i.e. energy provided by oxygen consumption). However, Aftalion and Bonnans (2014) remark that what he encompasses in the balance is the accumulated oxygen deficit: $\mathrm{e}^0 - e(t)$. Therefore, $e(t)$ in Eq. (1) is in fact the anaerobic energy (i.e. energy provided by glycogen and lactate). In order to reproduce the results of (Hanon et al. 2008, 2010; Hanon and Thomas 2011), the oxygen uptake σ introduced in (Aftalion and Bonnans 2014) is piecewise defined: in the most part of the race, σ is constant equal to its maximal value σ_max, but it is increasing at the beginning of the race, and decreasing at the end (see Fig. 1). In fact, σ depends on five parameters: the initial value at rest σ_r, the maximal value σ_max, the final value σ_f and two parameters φ and e_cr which denote the transition point from one zone to another, $\varphi, e_\mathrm{cr} \in (0, 1)$:

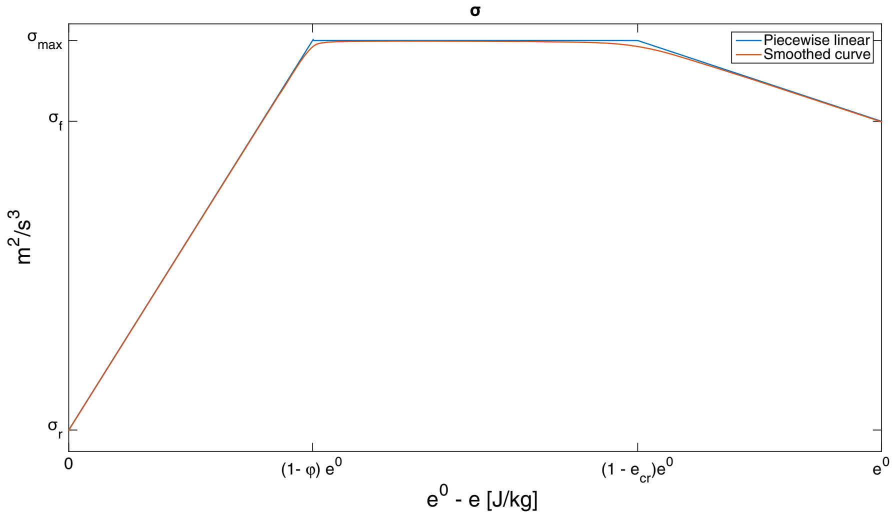

Fig. 1 Typical curve σ versus $(\mathrm{e}^0 - e)$ for a 1500 m race

$$\sigma(e; \sigma_{\max}, \sigma_{\mathrm{f}}, \sigma_{\mathrm{r}}, \varphi, e_{\mathrm{cr}}) = \begin{cases} \sigma_{\max} \frac{e}{\mathrm{e}^0 e_{\mathrm{cr}}} + \sigma_{\mathrm{f}} \left(1 - \frac{e}{\mathrm{e}^0 e_{\mathrm{cr}}}\right) & \text{if } e < \mathrm{e}^0 e_{\mathrm{cr}} \\ \sigma_{\max} & \text{if } \mathrm{e}^0 e_{\mathrm{cr}} \leq e \leq \mathrm{e}^0 \varphi \\ \sigma_{\mathrm{r}} + \frac{(\sigma_{\max} - \sigma_{\mathrm{r}})(\mathrm{e}^0 - e)}{\mathrm{e}^0 (1-\varphi)} & \text{if } e \geq \mathrm{e}^0 \varphi, \end{cases} \tag{5}$$

The oxygen uptake σ as defined in (5) is continuous but not C^1. In the numerical simulations, it has been smoothed, since from the physiology, it is clear that the passage from one zone to the other occurs smoothly. The sigma used is shown in Fig. 1. Let us observe that, consistently with Hanon experimental results (Hanon et al. 2010), the functional form of σ does not change with the athlete, whilst the values of $\sigma_{\max}$ and σ_{f} do.

Moreover, in (Aftalion and Bonnans 2014), a second modification is introduced to the energy equation: for sufficiently long races (longer than 1000 m), it has been observed that slowing down recreates some of the anaerobic energy. Therefore, the energy equation results in:

$$\dot{e} = \sigma(e) + \eta(\dot{v}) - f v,$$

where η depends on the acceleration $\dot{v}$ and has the following form:

$$\eta(\dot{v}) = \begin{cases} 0 & \text{if } \dot{v} > 0 \\ c_\eta |\dot{v}|^2 & \text{if } \dot{v} \leq 0, \end{cases} \tag{6}$$

where c_η is a constant to be tuned. This leads to oscillations in the velocity profile (cf. Aftalion and Bonnans 2014, Fig. 2.4). In this paper, we will not focus on the term η and on the causes of the oscillations: we will briefly present some numerical results obtained with the term η in one simple case. For further details, see Aftalion and Bonnans (2014).

1.2 Equations for Two Runners

In real races, the competition between runners has a fundamental impact on the strategy. Starting from the works of Keller (1974), Quinn (2004) and Kyle (1979), Pitcher introduces a two-runners model in (Pitcher 2009) based on the slipstream. The observation is that running behind someone can save 1 or 2 s per lap in middle-distance races. Therefore, in the equations, the friction gets reduced when runners are just behind their competitor.

Some of the quantities in this model have a subscript i, which refers to the runner i: therefore x_i, v_i and e_i are, respectively, the position, the velocity and the energy of runner-i. All the physiological parameters which depend on the runner have the subscript i, too. Finally, the state variable x_D represents the distance between the runners, defined as $x_\mathrm{D} := x_2 - x_1$. The energy balance for each runner is the same as in (1), with a constant value of σ, while the dynamics equation incorporates an aerodynamical term. Because it is the relative position which is important, instead of using x_i, the position of each runner as parameters, we use x_1 and the relative position x_D. The resulting model is the following: for $i = 1, 2$

$$\begin{cases} \dot{x}_1 = v_1 & x_1(0) = 0 \\ \dot{x}_D = v_2 - v_1 & x_\mathrm{D}(0) = 0 \\ \dot{v_1} = f_1 - \frac{v_1}{\tau_1} - c_1 v_1^2 \left(1 - \gamma \left(\mathrm{e}^{-\alpha(x_\mathrm{D}-\beta)^2}\right)\right) & v_1(0) = 0 \\ \dot{v_2} = f_2 - \frac{v_2}{\tau_2} - c_2 v_2^2 \left(1 - \gamma \left(\mathrm{e}^{-\alpha(x_\mathrm{D}+\beta)^2}\right)\right) & v_2(0) = 0 \\ \dot{e_i} = \sigma_i - f_i v_i & e_i(0) = \mathrm{e}_i^0. \end{cases} \tag{7}$$

As in Keller's model, the velocities and energies equations are normalized with respect to the mass of the runner. The term $-c_i v_i^2$ is the friction with the air. It is necessary to highlight and separate the effect of the friction with the air from the other ones, because it is the only one that is reduced while running in the slipstream of someone else. This frictional term is modulated by $1-\gamma(\mathrm{e}^{-\alpha(x_\mathrm{D}\pm\beta)^2})$, which is shown in Fig. 2.

The parameter β represents the optimal distance a runner should keep from the other in order to obtain the maximal reduction of the air friction, while $\gamma \in (0, 1)$ is the percentage reduction at the optimal distance. The parameter α is an index of the variance of the phenomenon and is chosen large enough so that the exponential term becomes negligible as we deviate by half a metre from β. We observe that the choice of a non-symmetric term would probably be more accurate and realistic, but more complicated and with more parameter to estimate: for these reasons, in this work we use the term suggested by Pitcher (2009), which gives reasonable results when compared to real races. The parameters α, β and γ do not depend on the runner, however the parameter c is related to the drag coefficient and depends on the shape and surface properties of the athlete's body (see Quinn (2004) for further details). For simplicity, in this work, we consider $c_1 = c_2 = c$. If ever the athletes have different masses, then we would have $m_1 c_1 = m_2 c_2$.

The problem has physiological constraints:

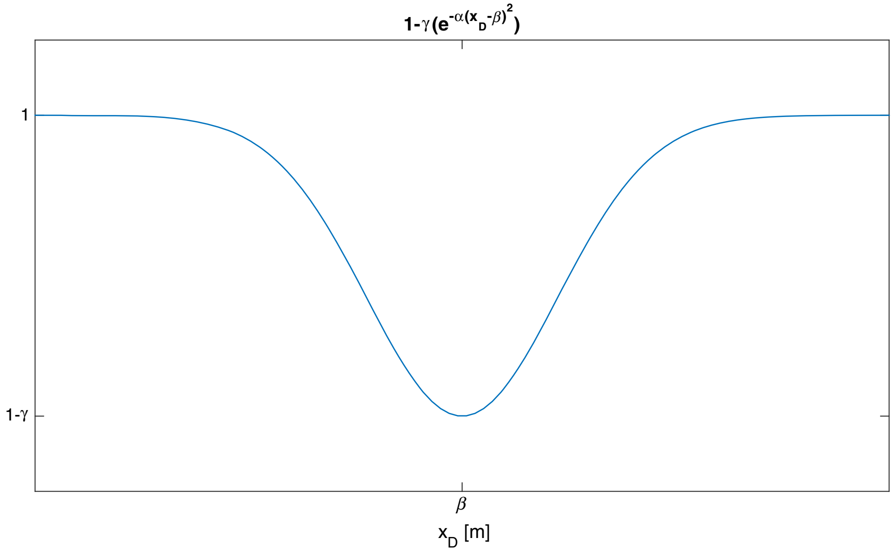

Fig. 2 Term that modulates the friction with the air in the two-runners model

$$e_i(t) \geq 0 \quad \forall t \geq 0, \; i = 1, 2. \tag{8}$$

In order to have the state equations for both runners defined on the same time interval, Pitcher chooses a fixed final time T. Moreover, she fixes the running strategy of runner 1 (therefore $f_1(\mathbf{t})$ is given), as the optimal strategy they would adopt if running alone, therefore the only control is $f_2(t)$. Formally, the resulting optimization problem is:

$$\max_{f_2 \in \mathcal{F}} x_D(T) \quad \text{s.t.} \quad (7)\text{–}(8). \tag{9}$$

It is clear that this strategy is not realistic, because all the runners adapt their strategy according to the performances of their opponents. In fact, one of the crucial point in a two-runners problem is how to model competition between them: some possible ways are game theory or multi-objective optimization. However, in this work, as explained below, we model the competition by leaving both strategies free, therefore having two controls, f_1 and f_2, and by encompassing in the cost functional the distance between the runners at the final time.

2 Mathematical Model

In this work, we use Pitcher's two-runners model (Pitcher 2009) and the single-runner model of Aftalion and Bonnans (2014) to build a new model for two runners, which incorporate psychological factors.

In order to simplify the notation, we substitute the expression $\sigma(e_i; \sigma_{\max,i}, \sigma_{\mathrm{f},i}, \sigma_{\mathrm{r},i}, \varphi_i, e_{\mathrm{cr},i})$ from (5) with $\sigma_i(e_i)$. We recall that the friction in Pitcher's paper is

$-cv_i^2\left(1-\gamma e^{-\alpha(x_D\pm\beta)^2}\right)$ and is supposed to model the slipstream. Shielding behind someone has a strong impact when there is wind or when the velocity is high, however in real races this position does not come entirely from slipstream but from strategic factors, too, for which the position 1 m behind is the best. Therefore, the potential $1-\gamma e^{-\alpha(x_D\pm\beta)^2}$ can also be considered to model a psychological factor which consists in trying to follow one's competitor, in order to be able to overtake. Indeed, it is a potential which has a minimum at distance β behind and decreases global friction. On the other hand, when the other runner is too far, there is no benefit. Let us observe that this model does not take into account the lateral displacement, and therefore the additional propulsive force, that is necessary to overtake. One can model the fact that overtaking requires some additional energy, by possibly using a non-symmetric potential well $1-\gamma e^{-\alpha(x_D\pm\beta)^2}$, with a varying α, which is not what we have done in the simulations presented to reduce the number of parameters involved. We obtain the following equations: for $i=1,2$

$$\begin{cases} \dot{x}_1 = v_1 & x_1(0)=0 \\ \dot{x}_D = v_2 - v_1 & x_D(0)=0 \\ \dot{v_1} = f_1 - \frac{v_1}{\tau_1} - cv_1^2\left(1-\gamma\left(e^{-\alpha(x_D-\beta)^2}\right)\right) & v_1(0)=0 \\ \dot{v_2} = f_2 - \frac{v_2}{\tau_2} - cv_2^2\left(1-\gamma\left(e^{-\alpha(x_D+\beta)^2}\right)\right) & v_2(0)=0 \\ \dot{e_i} = \sigma_i(e_i) + \eta_i(\dot{v_i}) - f_i v_i & e_i(0)=e_i^0. \end{cases} \tag{10}$$

The Eq. (10) are defined for $t\in(0,T)$, where T is the time at which the first of the two runners reaches the final distance D; to model this, it is necessary to add the following boundary condition to the system:

$$(x_1(T)-D)(x_2(T)-D)=0. \tag{11}$$

As in the previous models, the energy has a lower bound:

$$e_i(t)\geq 0 \quad \forall t\in(0,T),\ i=1,2. \tag{12}$$

The choice of the cost functional, i.e. the quantity to be minimized, is a key point. As said before, in contrast with Pitcher's choice, in this case none of the strategies is fixed, therefore there are two controls: $f_1(t)$ and $f_2(t)$. Here, we propose to minimize the following quantity, given a proper constant weight $c_w>0$:

$$J(f_1,f_2)=T+c_w|x_D(T)|. \tag{13}$$

The aim of this choice is to minimize the final time of the winner, and the term $c_w|x_D(T)|$ models the fact that the loser has tried to win as well. Different values of c_w can lead to different results, as in real races when two runners compete against each other multiple times the outcome of the race can change.

The resulting problem is:

$$\min_{f_i \in \mathfrak{F}_i} J \quad \text{s.t.} \quad (10)\text{–}(11)\text{–}(12), \tag{14}$$

where $\mathfrak{F}_i$ is the set of the admissible controls, and depends on the athlete. For physiological reasons, it is necessary to impose a bound to the variations of $\dot{f}$, in addition to the bounds on f already introduced, related to the fact that athletes cannot vary their propulsive force too quickly (see more details in Aftalion and Bonnans 2014). This leads to the following definition of $\mathfrak{F}_i$:

$$\mathfrak{F}_i := \left\{ f : 0 \leq f(t) \leq f_{\mathrm{M},i},\ |\dot{f}(t)| \leq K_i \ \forall t \in (0, T) \right\}, \tag{15}$$

where K_i and $f_{\mathrm{M},i}$ are constants depending on the athlete, which model the fact that every runner has a limited maximal force ($f_{\mathrm{M},i}$) and cannot vary it too quickly (K_i). The rest of the paper consists in providing numerical simulations of (14).

3 Numerical Results

All the results presented in this section are obtained with the free software BOCOP Bonnans et al. (2014). The equations are solved with a finite difference scheme (implicit Euler), while the optimization problem is solved with an iterative method, using as stopping criterion the difference between successive iterates, with a tolerance of 10^{-10}.

The aim of these simulations is to find out if a runner can win a race against someone stronger, by running behind the first part of the race, and to quantify in term of variation of some parameters how much weaker they can be and when the best time to overtake is.

The single-runner strategy has been mathematically proven and numerically computed in (Aftalion and Bonnans 2014), and found experimentally in (Hanon et al. 2008), and it consists in three parts:

- a first part of maximal force with a strong acceleration during which the peak velocity is achieved,
- a second part in which the propulsive force first decreases smoothly and then increases again, with the corresponding decrease and increase in the velocity,
- a final part at maximal force and maximal velocity again, until zero energy level is reached, where the velocity drops.

From the two-runners model, we expect an overall similar strategy: however, the additional term in the velocities equations encourages one of the runners to start slightly slower and to position themselves at distance β from the other. This allows them to keep the same velocity as the other runner while using a smaller propulsive force, which, in turn, leads to a lower energy consumption. We expect that, at a certain point during the race, the runner who is behind, will overtake the other by using the energy they have saved throughout the race and will be able to perform a longer final sprint. Moreover, it is reasonable to think that this moment occurs sooner if the

Table 1 Parameters values for 1500 m

Parameter	Unit of measurement	Value
τ	s	1.33
c	m^{-1}	0.0028
e^0	J/kg	1400
σ_r	m^2/s^3	6
σ_{max}	m^2/s^3	24.22
σ_f	m^2/s^3	20.44
φ	–	0.5
e_{cr}	–	0.3
c_η	s	4
α	m^{-2}	10
β	m	1
γ	–	0.8
f_M	N/kg	5
c_w	–	0.1

runner who runs the first part of the race behind is stronger. Let us observe that this running strategy could also lead, in some favourable situations, to an improvement of the personal record. Nonetheless, it is important to underline that the decrease in the final time is not the main goal for a runner in some occasions, such as the Olympic finals, in which the final position is definitely more important than the final time: in Thiel et al. (2012), study, starting from the Beijing 2008 Olympic games data and the world records data, how the difference in the goal affects the pacing strategies: win the race versus minimizing the final time.

How to estimate the parameters values starting from a race is beyond the scope of this paper. For this reason, the reference values for the parameters used in the simulations are taken from the literature and reported in Table 1. The initial, maximal and final value of sigma (respectively, σ_r, σ_{max} and σ_f) are taken from the $\dot{V}O_2$ values reported in Hanon et al. (2008): let us observe that these values are given in ml kg^{-1} min^{-1}. In order to convert them in the unity of measurement needed (i.e. $m^2\,s^{-3} = J\,kg^{-1}\,s^{-1}$), we consider that the uptake of 1 ml of oxygen is often converted into an energy expenditure estimate of 21 J. It is then necessary to convert the minutes in seconds, obtaining in this way the conversion factor 21/60. The values chosen for φ and e_{cr} aim at fitting the $_2$ profile reported in Fig. 2 of Hanon et al. (2008). The values for f_M and τ are strongly related: in fact, a first-order approximation of the maximal velocity v_{peak} a runner can reach is τf_M. Therefore, starting from the velocity profile plotted in Fig. 1 of Hanon et al. (2008), one can chose a couple of reasonable values. The value of the constant c is taken from Quinn (2004), c_η from Aftalion and Bonnans (2014), α and β from Pitcher (2009) and, finally, γ from Pugh (1971).

First of all, let us consider the perfectly symmetric situation, in which the two runners have the same parameters. In this case, it is very influential which runner is running the first part of the race behind: this can be mathematically modelled by

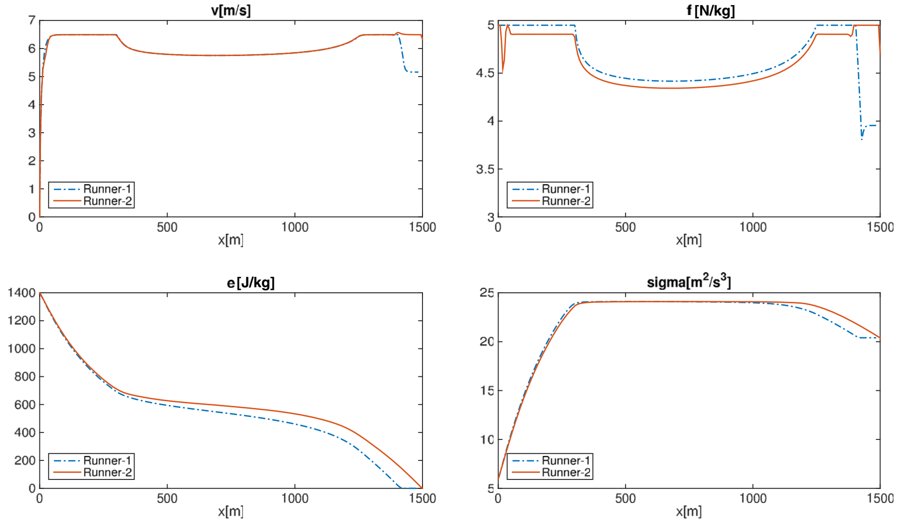

Fig. 3 Competition between two runners with the same parameters

choosing, in a proper way, the initial guess for the variable x_D given to the iterative method that solves the optimization problem. In fact, giving as initial guess β (or $-\beta$) forces runner-1 (or runner-2) to start the race more slowly and to position themselves behind for the first part. If one gives as initial guess $x_D = 0$, one finds a solution in which the runners overtake each other multiple times, which is not very realistic. The non-uniqueness of the solution is not surprising, especially in a perfectly symmetric situation, such as the one considered: if a certain couple of strategies $\{f_1(t), f_2(t)\}$ provides a minimum for the cost functional, the couple $\{f_2(t), f_1(t)\}$ provides a minimum, too. This can be considered to model the fact that if the same two runners run against each other multiple times, the outcome of the race can change. The results of this simulation are shown in Figs. 3 and 4: one can observe that the runner who stays behind can keep the same velocity as the other runner, while using a significantly lower propulsive force and therefore having a much lower energy consumption. All the graphs in Fig. 3 have the position on the x-axis, which means that the velocity of runner-i is plotted with respect to the position of runner-i (and it is the same for the propulsive force, the energy and sigma). The choice of plotting with respect to position, and not time, has been made because it is the most common in the sports literature. Figure 4 shows the distance between the runners: the overtaking occurs at about 94% of the race, corresponding to 1416 m. This variable is plotted with respect to a normalized time (i.e. t/T), because it does not concern only one runner, but both of them, therefore it does not make sense to plot it with respect to the position of any of them.

From this first result, it is clear that running behind someone for the most part of the race allows runners to win against athletes as strong as themselves. We now want to investigate how the strategy of runners changes if they are running alone or behind someone else. In Fig. 5, two performances of the same runner are compared: the blue line represents the optimal strategy of the runner running alone (they complete the

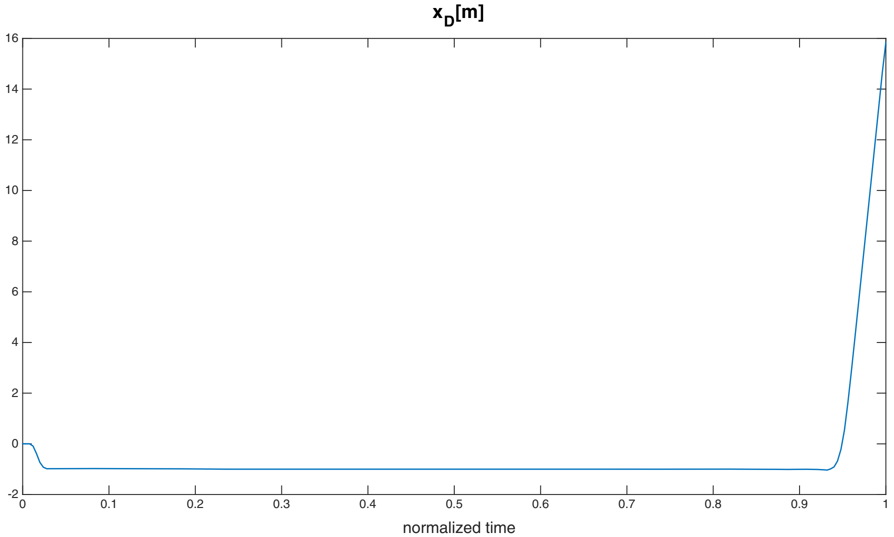

Fig. 4 Competition between two runners with the same parameters; distance between the runners

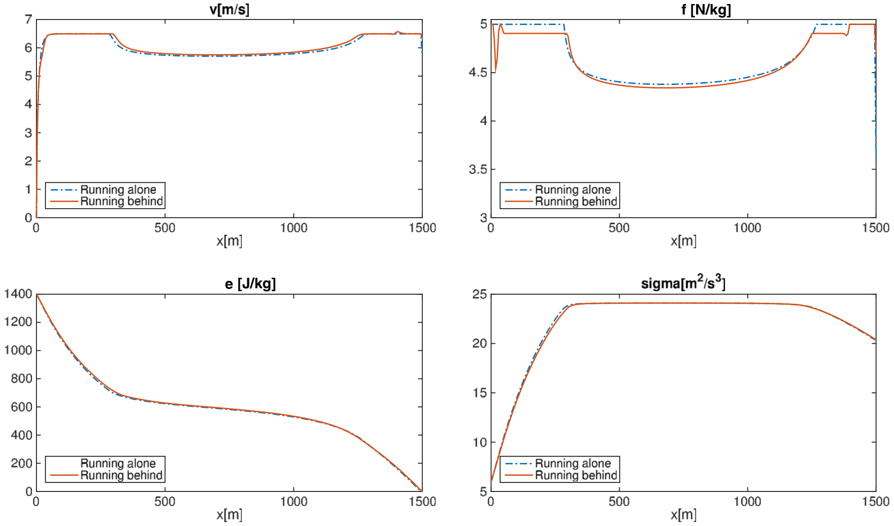

Fig. 5 Running alone versus running behind

race in 249.681 s), the red line represents the strategy adopted when running behind someone as strong as themselves (they completes the race in 247.822 s). This difference in the final time (almost 2 s of improvement) is equivalent to a difference of about 0.05 m/s in the mean velocity.

However, what we have simulated is not the most favourable situation to reduce the final time, and therefore to improve the mean velocity. If the aim is exclusively to improve the personal best performance and not to win the race, the best scenario

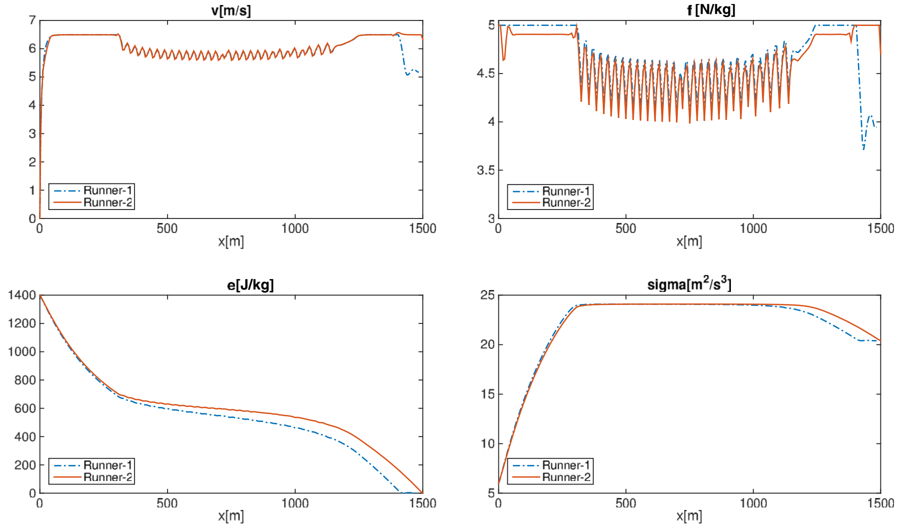

Fig. 6 Competition between two runners with the same parameters, with the recreation term η

possible for a runner is to run behind someone slightly stronger for the whole race. However, let us observe that the opponent must not be too much stronger: in this case, the runner would use too much energy to stay behind and would soon reach the zero energy level, which would cause a drop in the propulsive force too soon in the race. Nevertheless, from Fig. 5 we can still observe how running behind someone else allows an athlete to have a higher velocity in spite of keeping a smaller force throughout the race.

The results presented in Figs. 3, 4 and 5 are obtained with the model (10) without the recreational term η introduced in (6). In Figs. 6 and 7 we present the same scenario, but with recreation: the two runners have the same parameters. We recall that, as explained in Sect. 1.1, the recreational term η leads to oscillations in the velocity profile. One can observe, comparing Fig. 3 with Fig. 6 and Fig. 4 with Fig. 7, that the strategy does not change: runner-2 slows down at the beginning, in order to be behind runner-1; in the middle part of the race the propulsive forces are oscillating (this behaviour is caused by the additional term η and can be found also in the single-runner problem, see Aftalion and Bonnans (2014) for further details), and the mean value around which the propulsive force of runner-1 is oscillating is slightly bigger than the one around which the force of runner-2 is oscillating; the energy curve does not change significantly, compared to the previous results. From Fig. 7, one can notice that runner-2 overtakes at about 94.85% of the race, which corresponds to 1417 m: 1 m later, if compared with the case without oscillations. Let us observe that in this case the final time is smaller: in fact runner-2 completes the race in 247.75 s. This decrease in the final time, when adding the recreation η, is consistent with the results for the single-runner problem presented in (Aftalion and Bonnans 2014).

Finally, we can say that the recreation term does not change the strategy of the race, however the results are more difficult to read, due to the oscillations. For this reason, from now on we will present only results obtained without η.

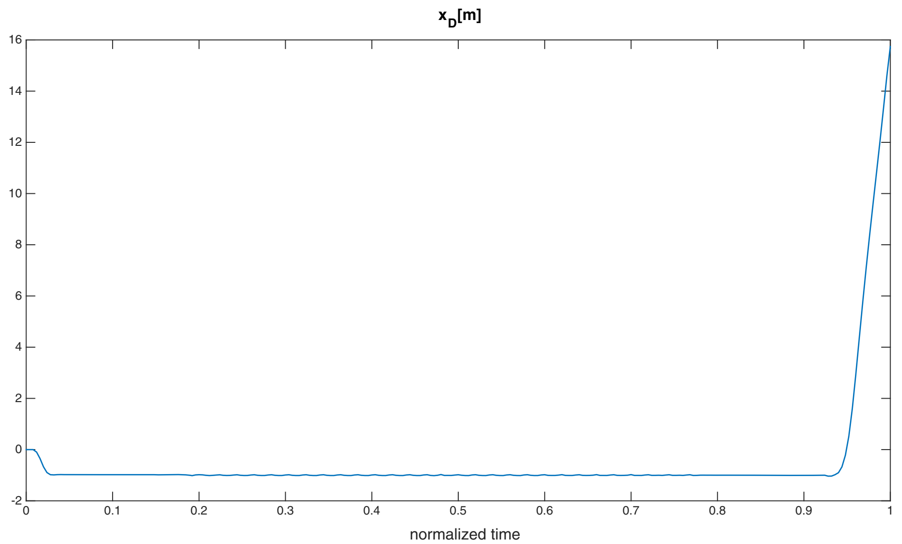

Fig. 7 Competition between two runners with the same parameters, with the recreation term η; distance between the runners

We now want to find the threshold values, i.e. how much runners can be weaker than their opponent and still win the race by running behind. Therefore, we vary one parameter at a time, making runner-2 weaker. Figures 8 and 9 show the results for two runners who have a different initial energy. Given the reference value for $e_1^0 = 1400$ (as in Table 1), the lowest initial energy runner-2 can have, while still being able to win the race, is:

$$e_2^0 = 1275\,\text{J/kg}.$$

At the beginning of the race, runner-2 slows down in order to stay behind: in this way runner-2 manages to keep the same velocity as runner-1 using a smaller force; this leads to a smaller energy consumption, therefore at the end of the race runner-2 has enough energy to speed up and overtake runner-1. For the boundary condition (11), the time stops as soon as the first runner finishes the race. Therefore, when runner-2 reaches the finish line (i.e. 1500 m), runner-1 has covered only 1498.13 m. The final time of runner-2 is 249.43 s, while their best performance running alone is 251.403 s, again an improvement of almost 2 s. As shown in Fig. 9, the overtaking occurs later in the race, if compared with the case in which the runners were equally strong: here occurs at 99% of the race (i.e. about 1487 m). This is reasonable: in fact, being weaker, runner-2 has to exploit the advantage of staying behind as long as possible.

In Figs. 10 and 11, the two runners have different oxygen uptake. The threshold values are the following:

$$\sigma_{\max,1} = 24.22\,\text{m}^2/\text{s}^3, \quad \sigma_{f,1} = 20.44\,\text{m}^2/\text{s}^3$$
$$\sigma_{\max,2} = 23.75\,\text{m}^2/\text{s}^3, \quad \sigma_{f,2} = 20.18\,\text{m}^2/\text{s}^3.$$

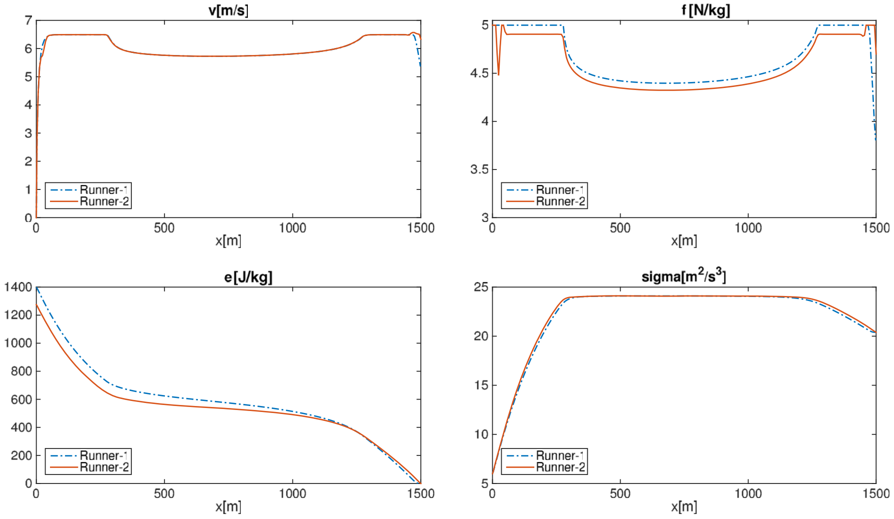

Fig. 8 Competition between two runners with different e^0

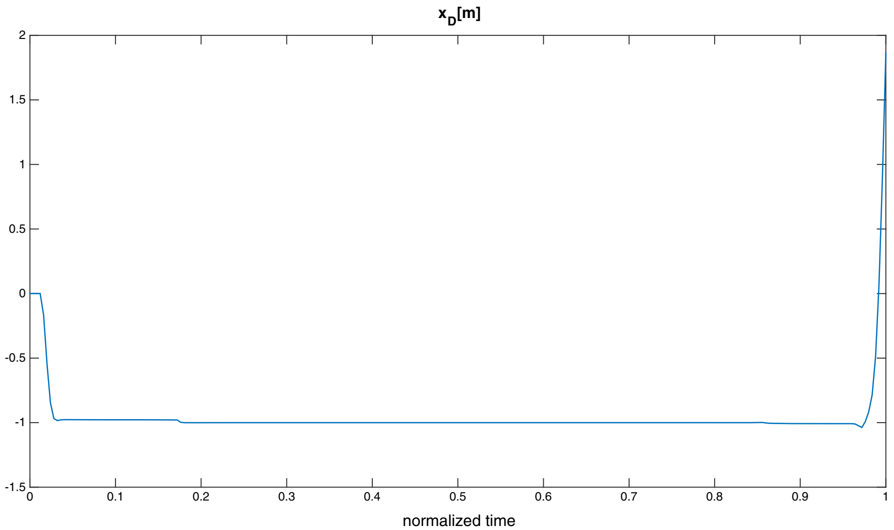

Fig. 9 Competition between two runners with different e^0; distance between the runners

The initial strategy is the same as the previous case: runner-2 slows down in order to stay behind; this allows them to keep the same velocity as runner-1 using a smaller force and therefore compensating the smaller σ. The speed up in the final part is less evident than it was in the previous case, because the difference between the energies is smaller. The final distance covered by runner-1 in this case is 1499.72 m. The final time of runner-2 is 249.66 s, compared to 251.665 s if running alone. Figure 11 shows the distance between the runners during the race: as in the previous case, the overtaking occurs late in the race.

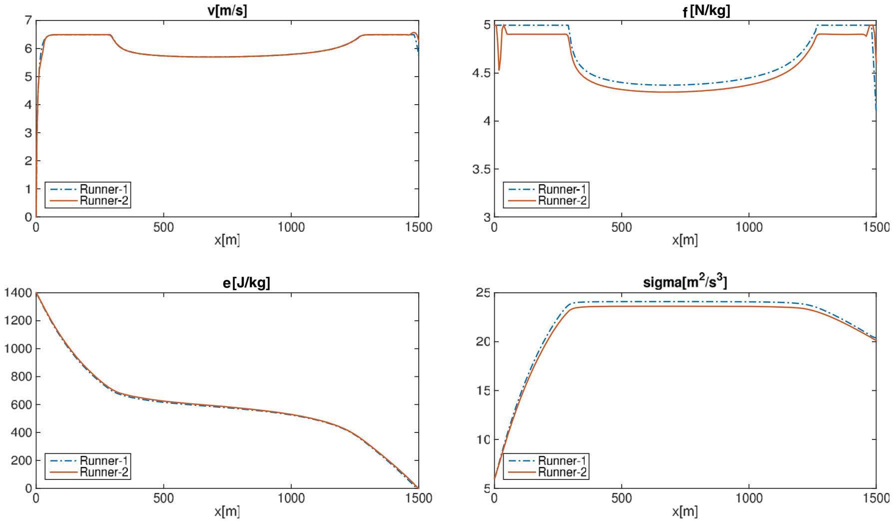

Fig. 10 Competition between two runners with different σ

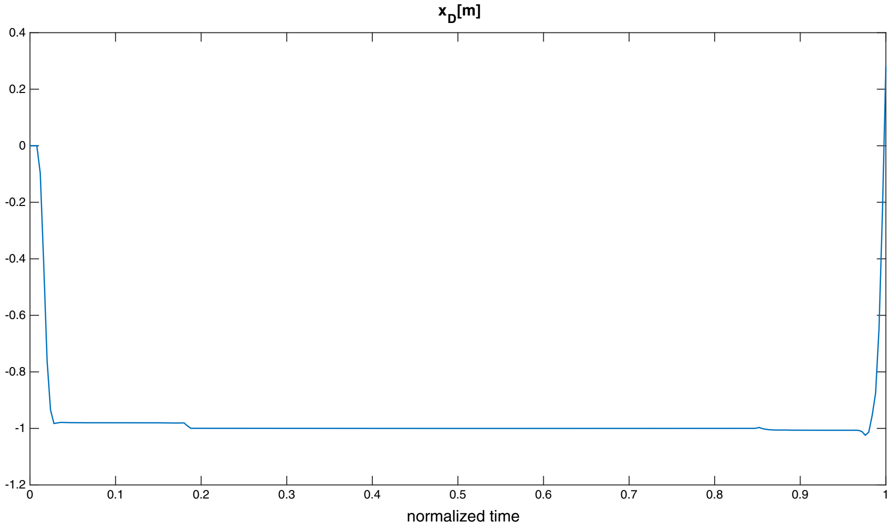

Fig. 11 Competition between two runners with different σ; distance between the runners

Figures 12 and 13 show the results for two runners who have a different value for τ. The threshold values are:

$$\tau_1 = 1.33\,\text{s} \quad \tau_2 = 1.31\,\text{s}.$$

A smaller τ indicates a bigger drop in velocity due to frictional effects, therefore, a greater force is necessary to keep the same velocity. Being behind another runner is a way to compensate this weakness: as shown in Fig. 12, runner-2 manages to keep the

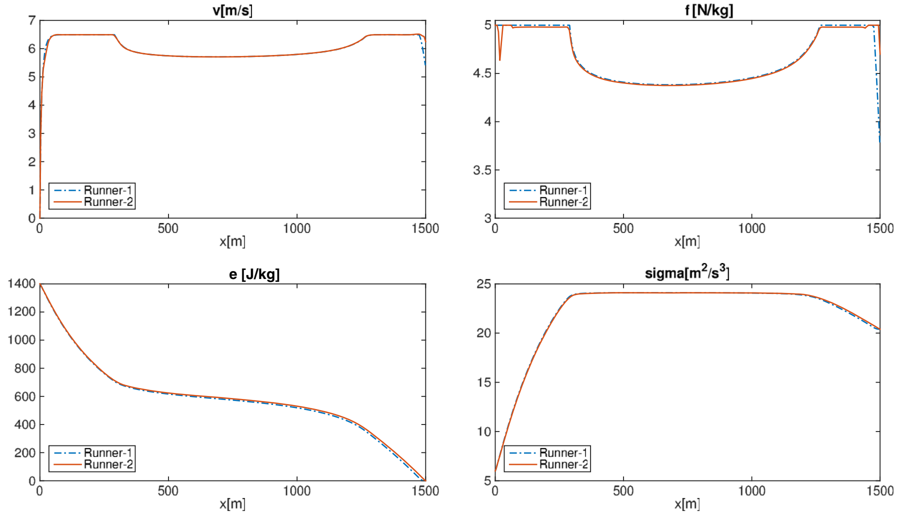

Fig. 12 Competition between two runners with different τ

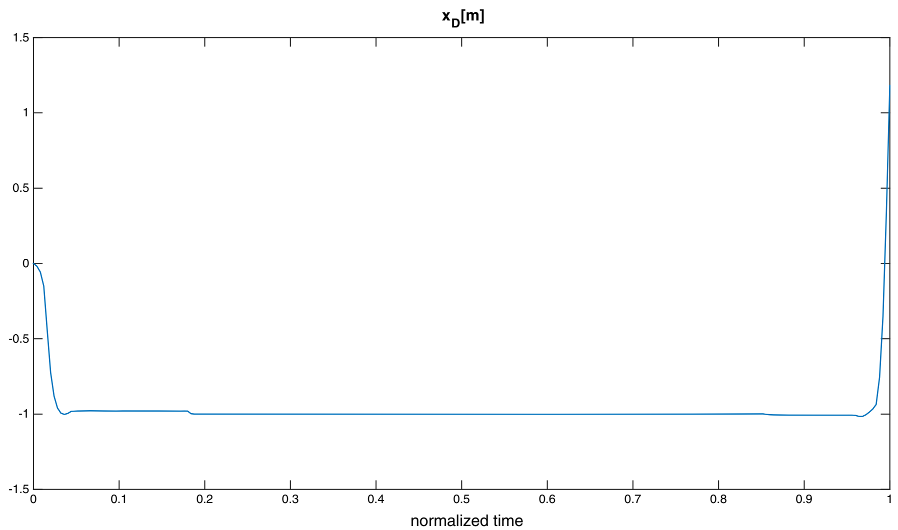

Fig. 13 Competition between two runners with different τ; distance between the runners

same velocity as runner-1, using a slightly smaller force, in spite of having a smaller τ. In the final part, when the difference in the energies is sufficiently big, runner-2 overtakes runner-1. In this case, at the end of the race, runner-1 has covered a distance of 1498.82 m. The final time of runner-2 is 249.536 s, while their best performance running alone is 251.83 s, i.e. they have an improvement of more than 2 s.

Finally, let us consider a case in which the runner who starts behind is stronger. For this purpose, we use the following parameters:

$$\tau_1 = 1.29\,\text{s} \qquad \tau_2 = 1.33\,\text{s}.$$

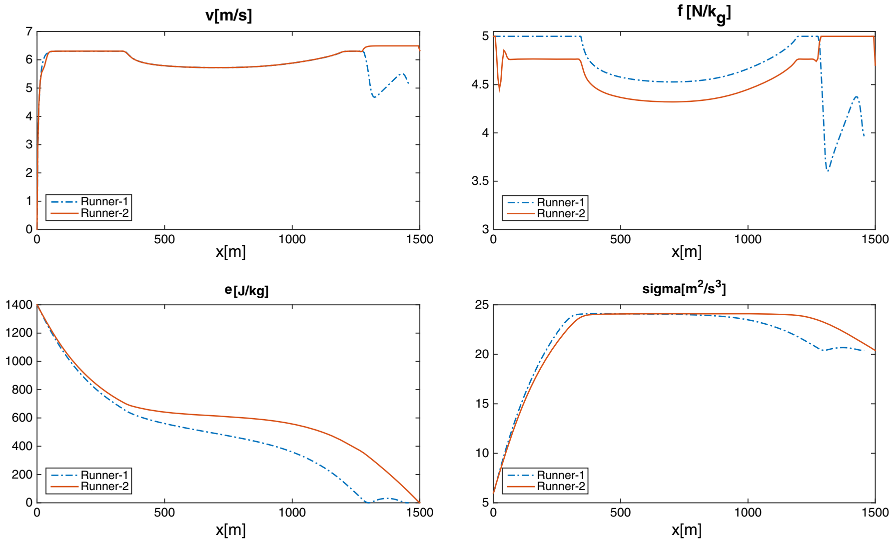

Fig. 14 Competition between two runners with different τ; stronger runs the first part of the race behind

All the other parameters remain unvaried (reference values from Table 1). Figures 14 and 15 show the results obtained: in this case, the difference between the athletes is much bigger than in the previous ones, and this is evident from all the curves in Fig. 14. From Fig. 15, one can notice that the overtaking occurs very early in the race: at about 87.1%, i.e. 1290 m. What is interesting in this case is that runner-2 completes the race in 248.726 s, which is almost 1 s less than their best performance running alone: therefore, in order to improve a personal record, it is not necessary to run behind someone stronger.

Let us now compare the results obtained here with Pitcher's ones. In Pitcher (2009), Fig. 5.2, when the weaker runner is the one with the fixed strategy, the stronger runner remains only slightly ahead of their opponent until nearly the end of the race. This is in order that the weaker runner does not gain the advantage of running in the slipstream of the stronger for a long part of the race. However, it is the runner who stays behind who should adjust their position with respect to the other one, and not the opposite. The advantage of having two strategies free allow us to avoid this unrealistic result, and in this case, we get that the weaker runner stays 1 m behind, and either wins the race if the difference in energy is not too big or drops following if they do not have enough energy.

Finally, we want to analyse the strategy and to see when the overtaking occurs in real races and compare them with our results. For this purpose, we have considered the men's 1500 m finals of three different competitions: Beijing 2008 Olympic games, Rome 2014 IAAF Diamond League and Singapore 2015 SEA Games. Videos of the races can be found on the internet (YouTube Channel 2008, 2014, 2015). We want to point out that the athlete who won the Beijing 2008 Olympics was disqualified 1 year later for doping and his gold medal was reassigned. Nonetheless, it is still interesting to analyse the race, knowing that doping increases the maximal value of σ but delays

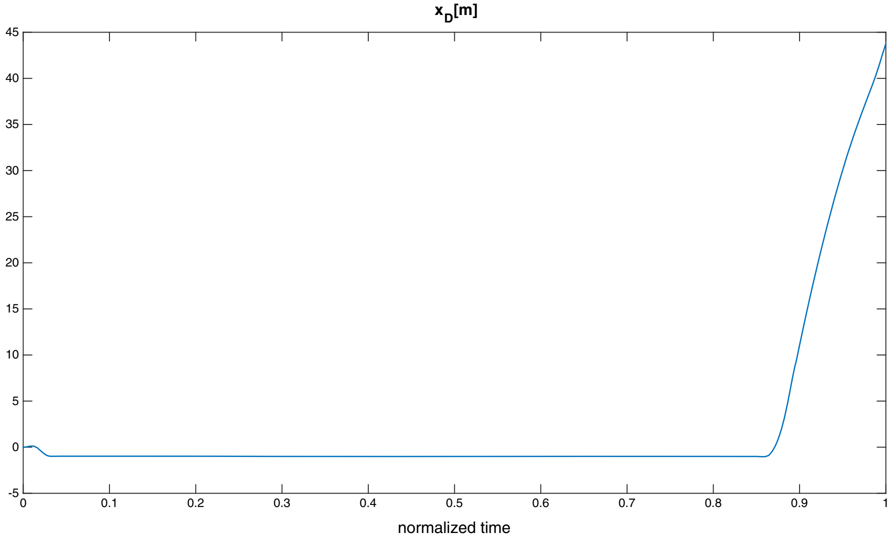

Fig. 15 Competition between two runners with different τ; stronger runs the first part of the race behind; distance between the runners

the time at which the peak velocity is reached: this should lead to a slower start, but it provides a capacity to keep a higher velocity for a longer part of the race. From the three videos (YouTube Channel 2008, 2014, 2015), one can observe that the winner always runs the first part of the race behind, and this is consistent with our numerical results. The overtaking occurs at 84.6% of the race in Beijing 2008, at 96.9% in Rome 2014 and at 91.8% in Singapore 2015: these values are close to our numerical results, that vary between 87 and 99% of the race depending on the difference between the athletes.

4 Conclusion

In this work, we have presented a new model for a two-runners problem, starting from the single-runner model of Aftalion and Bonnans (2014) and from the two-runners model of Pitcher (2009), changing the optimal control problem. The key of our simulations is that they quantify very precisely in terms of physiological parameters, optimal control problems and numerical simulations, phenomena which are only qualitatively understood. In this paper, we do not take into account the curvature of the track, which is the aim of an upcoming paper, since it requires more effort in the modelling.

As expected, going from one runner to two runners does not change the main characteristics of the velocity profile individuated already in (Aftalion and Bonnans 2014). We can still clearly distinguish the different phases of the race: the fast start, with maximal propulsive force and strong acceleration until the peak velocity is reached; an intermediate phase in which the propulsive force and the velocity first smoothly decrease and then increase; a final part at maximal force again, where the runner speeds up (final sprint), followed by a very short zero energy arc, in which there is a drop in force and velocity. Running behind someone allows to keep a velocity with a smaller propulsive force than the one needed when running alone; this leads to a smaller energy consumption, therefore the zero energy level occurs later and the speed up at the end of the race is more pronounced.

Our numerical results show that if a runner has a fast start and leads the race for the most part, even if they are slightly stronger than their opponent, at the end they are overtaken: in order to lead the race and win, the physiological difference between the athletes has to be significant. Furthermore, we have shown how runners can improve their personal best performance by exploiting the advantage of running behind someone else, who can be stronger or weaker. The most significant improvements are obtained by running behind someone stronger.

An interesting development, in order to have more realistic results, would be to include a delayed reaction term which takes into account the fact that runners cannot adapt instantaneously their strategy to changes in their competitor's strategy. This could be compared to a stochastic model. Finally, one could increase the number of runners, in order to be able to model real races more accurately. These considerations are outside the scope of this paper, but they can be important for future research.

This model suggests to use special runners to set the pace for others and help improve their racing times in training. The other major application for Olympic training could be for athletes to estimate whether they should stay behind or lead, and when, given their physiology, and that of their opponents, is the best time to overtake.

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