Milstein scheme for SQG under LU

Camilla Fiorini,
in collaboration with
Long Li, Étienne Mémin

6th STUOD Sandbox
February 12th, 2021

SQG system under location uncertainty

LU framework: based on the following decomposition of the Lagrangian velocity in two components

$$\mathrm{d}\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)\mathrm{d}t + \sigma(\mathbf{X}_t, t)\mathrm{d}\mathbf{B}_t$$

one can compute the **stochastic transport operator**:

$$\mathbb{D}_t b := \mathrm{d}_t b + \mathbf{v}^* \cdot \nabla b \ \mathrm{d}t + \sigma \mathrm{d}\mathbf{B}_t \cdot \nabla b - \frac{1}{2}\nabla \cdot (a \nabla b)\mathrm{d}t,$$

where

$$\mathbf{v}^* = \mathbf{u} - \frac{1}{2}\nabla \cdot a - \sigma(\nabla \cdot \sigma)$$

Therefore, the surface quasi geostrophic system under location uncertainty is:

$$\begin{cases} \mathbb{D}_t b = 0, \\ b = N(-\Delta)^{1/2}\psi, \\ \mathbf{u} = \nabla^{\perp}\psi, \end{cases}$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla \cdot (a\nabla b) - \mathbf{v}^* \cdot \nabla b \, \mathrm{d}s - \int_{t_0}^{t} \nabla b \cdot \sigma \mathrm{d}\mathbf{B}_s,$$

We model the noise by decomposing $\boldsymbol{\sigma}\mathrm{d}\mathbf{B}_t$ onto a basis, with two different approaches:

- POD approach $\boldsymbol{\sigma}\mathrm{d}\mathbf{B}_t \simeq \sum_{m=1}^{N} \varphi_m(\mathrm{x})\mathrm{d}\beta_t^m$
- SVD approach $\boldsymbol{\sigma}\mathrm{d}\mathbf{B}_t \simeq \sum_{m=1}^{N} \varphi_m(b_t)\mathrm{d}\beta_t^m$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla \cdot (a\nabla b) - \mathbf{v}^* \cdot \nabla b \, \mathrm{d}s - \int_{t_0}^{t} \nabla b \cdot \sigma \mathrm{d}\mathbf{B}_s,$$

$$\sigma \mathrm{d}\mathbf{B}_s \rightarrow \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We define the following functions:

$$f(b_t, t) = \frac{1}{2}\nabla \cdot (a\nabla b) - \mathbf{v}^* \cdot \nabla b \qquad g^m(b_t, t) = \nabla b \cdot \varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t, t) = f(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial s}(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

$$g^m(b_t, t) = g^m(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial g}{\partial s}^m(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g}{\partial b}^m(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g}{\partial b^2}^m(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \,\mathrm{d}s - \int_{t_0}^{t} \nabla b\cdot \sigma \mathrm{d}\mathbf{B}_s,$$

$$\sigma \mathrm{d}\mathbf{B}_s \to \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We define the following functions:

$$f(b_t,t) = \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \qquad g^m(b_t,t) = \nabla b\cdot\varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t,t) = f(b_{t_0},t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial f}{\partial s}}_{=0}(b_s,s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s,s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s,s)\mathrm{d}\langle b,b\rangle_s$$

$$g^m(b_t,t) = g^m(b_{t_0},t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial g^m}{\partial s}}_{=0}(b_s,s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g^m}{\partial b}(b_s,s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g^m}{\partial b^2}(b_s,s)\mathrm{d}\langle b,b\rangle_s$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla \cdot (a\nabla b) - \mathbf{v}^* \cdot \nabla b \, \mathrm{d}s - \int_{t_0}^{t} \nabla b \cdot \sigma \mathrm{d}\mathbf{B}_s,$$

$$\sigma \mathrm{d}\mathbf{B}_s \to \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We define the following functions:

$$f(b_t, t) = \frac{1}{2}\nabla \cdot (a\nabla b) - \mathbf{v}^* \cdot \nabla b \qquad g^m(b_t, t) = \nabla b \cdot \varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t, t) = f(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial s}(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

$$g^m(b_t, t) = g^m(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial g}{\partial s}^m(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g}{\partial b}^m(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g}{\partial b^2}^m(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

Towards the Milstein scheme

The bracket is:

$$\langle b, b\rangle_t = \langle \int_{t_0}^{\cdot} \sum_m g^m(b_s, s)\mathrm{d}\beta_s^m, \int_{t_0}^{\cdot} \sum_k g^k(b_\tau, \tau)\mathrm{d}\beta_\tau^k\rangle_t = \int_{t_0}^{t} \left(\sum_m g^m(b_s, s)\right)^2 \mathrm{d}s$$

As for the derivatives, we suppose to be in either one of the following cases:

- a and σ (therefore φ^m) do not depend on b and $\nabla \cdot \mathbf{v}^* = 0$
- a and σ depend on b and $\nabla \cdot \mathbf{v}^* = \nabla \cdot \nabla \cdot a = \nabla \cdot \sigma = 0$

Then we have:

$$\frac{\partial f}{\partial b}(\bar{b})b = f(b) \qquad \frac{\partial g^m}{\partial b}(\bar{b})b = g^m(b)$$

Milstein scheme

By replacing everything in the Itō formulas and then into the main equation, one finds:

$$b_t = b_{t_0} + f(b_{t_0})\Delta t - \sum_m g^m(b_{t_0})\Delta\beta^m + \int_{t_0}^{t}\int_{t_0}^{s} \sum_{m,k} g^m(g^k(b_\tau))\mathrm{d}\beta^k_\tau \mathrm{d}\beta^m_s \tag{1}$$

Euler-Maruyama

We define the following quantities:

$$G^{m,k} := g^m(g^k(b_{t_0})) \qquad I^{m,k} := \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}\beta^k_\tau \mathrm{d}\beta^m_s$$

Then the double integral in (1) can be approximated with:

$$= \Delta\beta^m\Delta\beta^k - \delta_{m,k}\Delta t$$

$$\sum_{m,k} G^{m,k} I^{m,k} = \sum_{m,k} G^{m,k}\frac{I^{m,k}+I^{k,m}}{2} + G^{m,k}\frac{I^{m,k}-I^{k,m}}{2}$$

Lévy area, which can be simulated

weak approximation

recursive (conditional) approximation

neglected

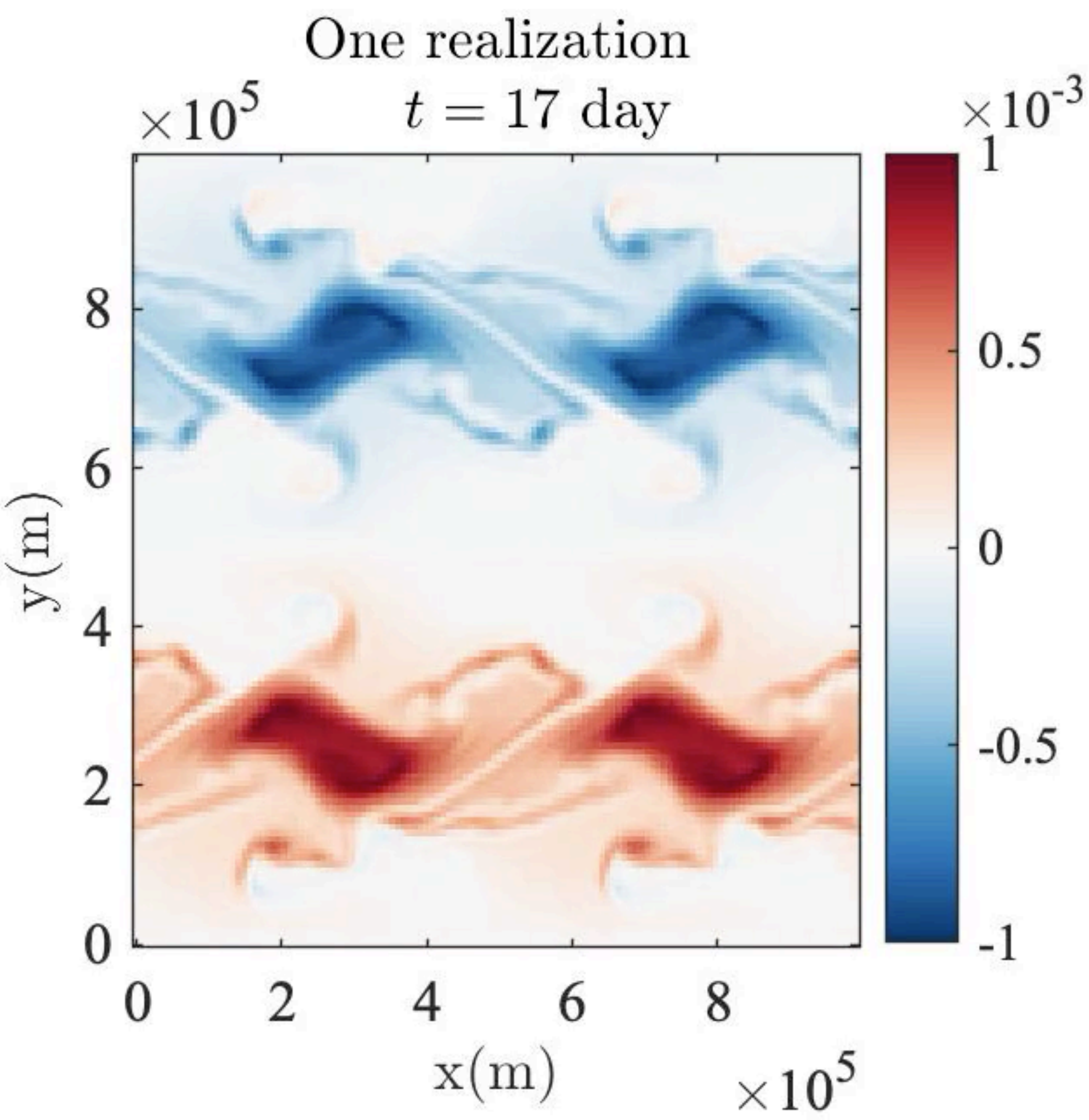
Remark: if G is symmetric (i.e. $G^{m,k} = G^{k,m}$), then the Lévy area is not necessary:

$$\sum_{m,k} G^{m,k} I^{m,k} = \frac{1}{2}\sum_{m,k} G^{m,k} I^{m,k} + G^{k,m} I^{k,m} = \sum_{m,k} G^{m,k}\frac{I^{m,k}+I^{k,m}}{2}$$

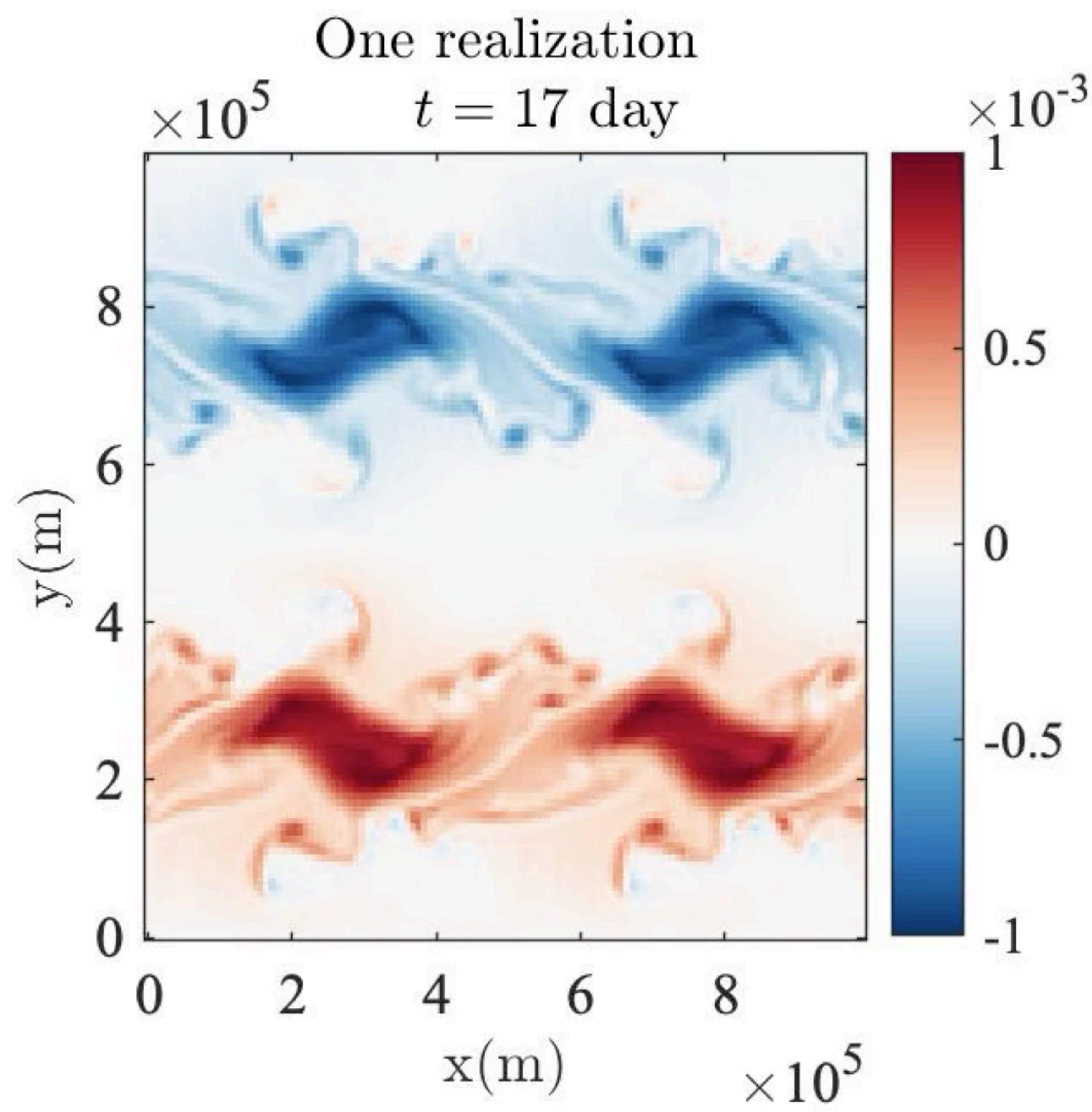
Numerical results

POD noise

Euler Maruyama

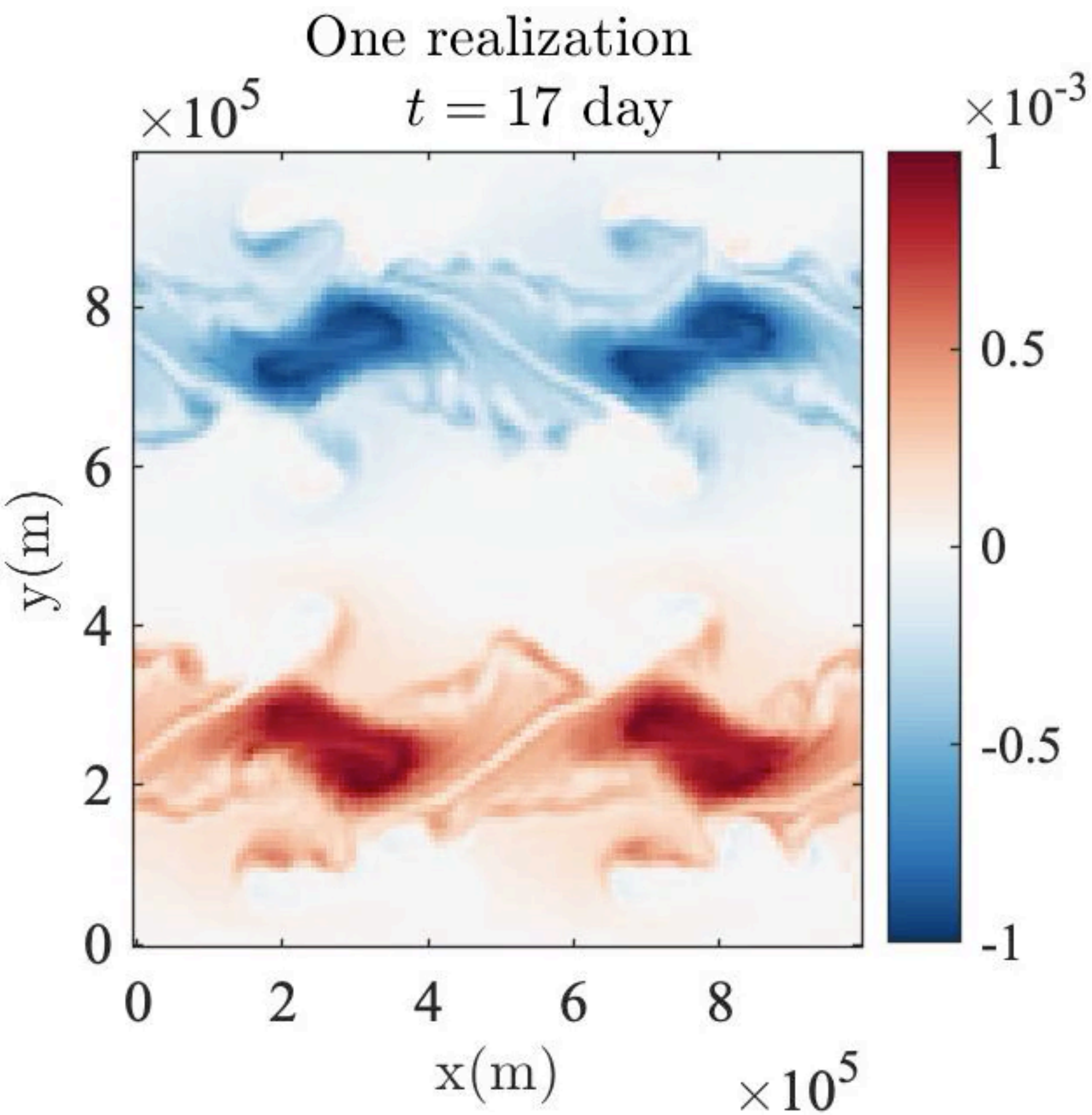


Milstein - weak

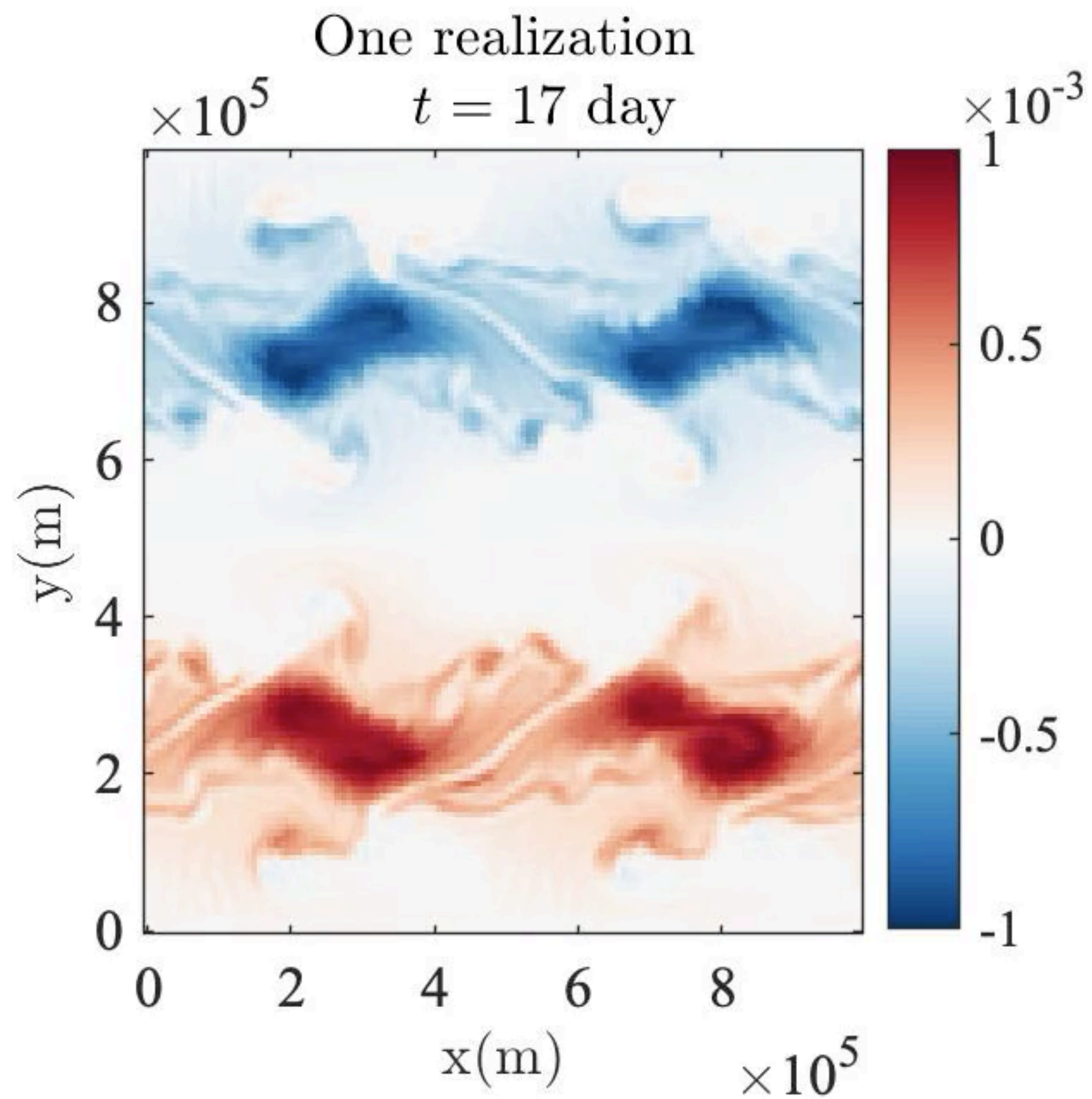


SVD noise

Euler Maruyama

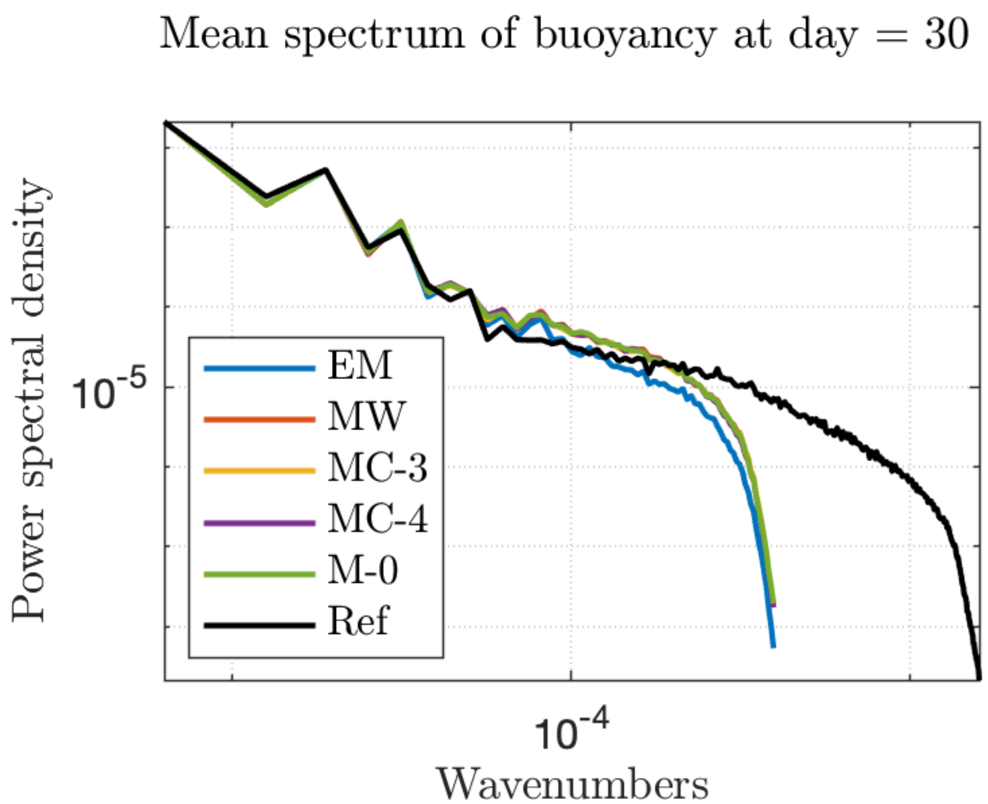


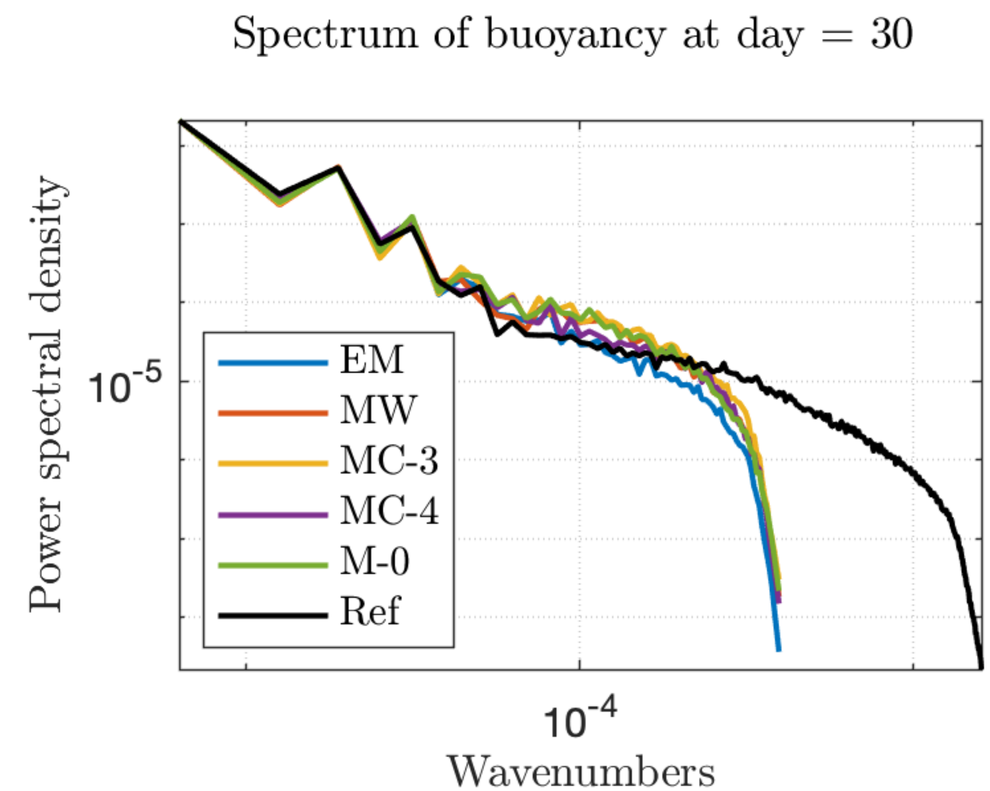
Milstein - weak



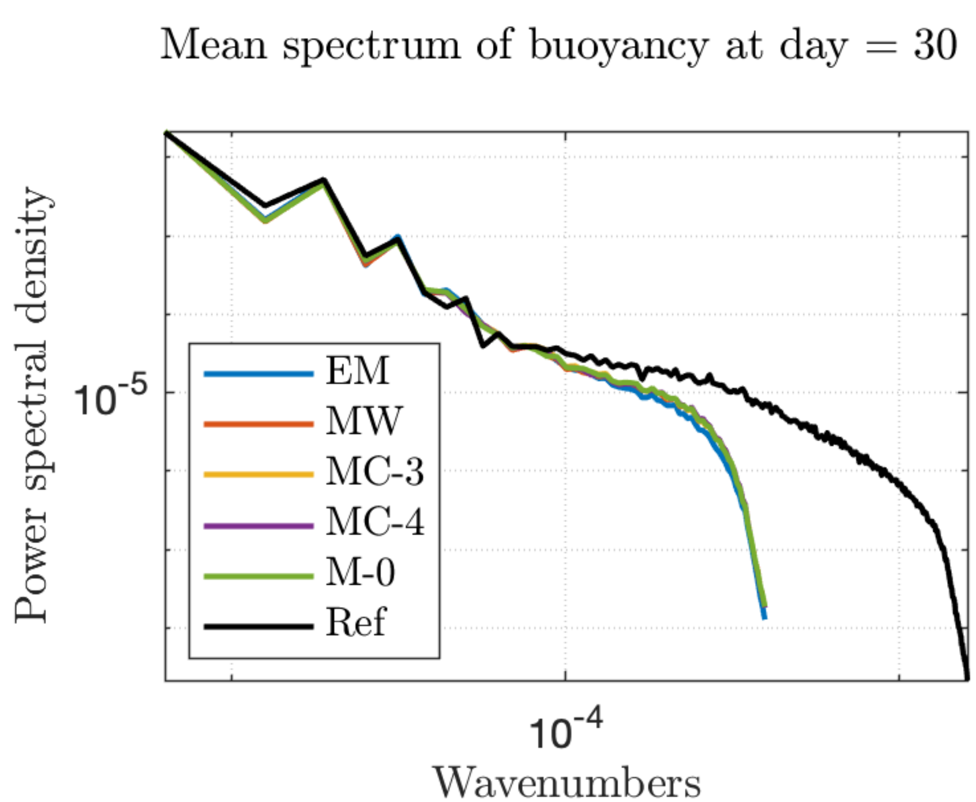
Numerical results

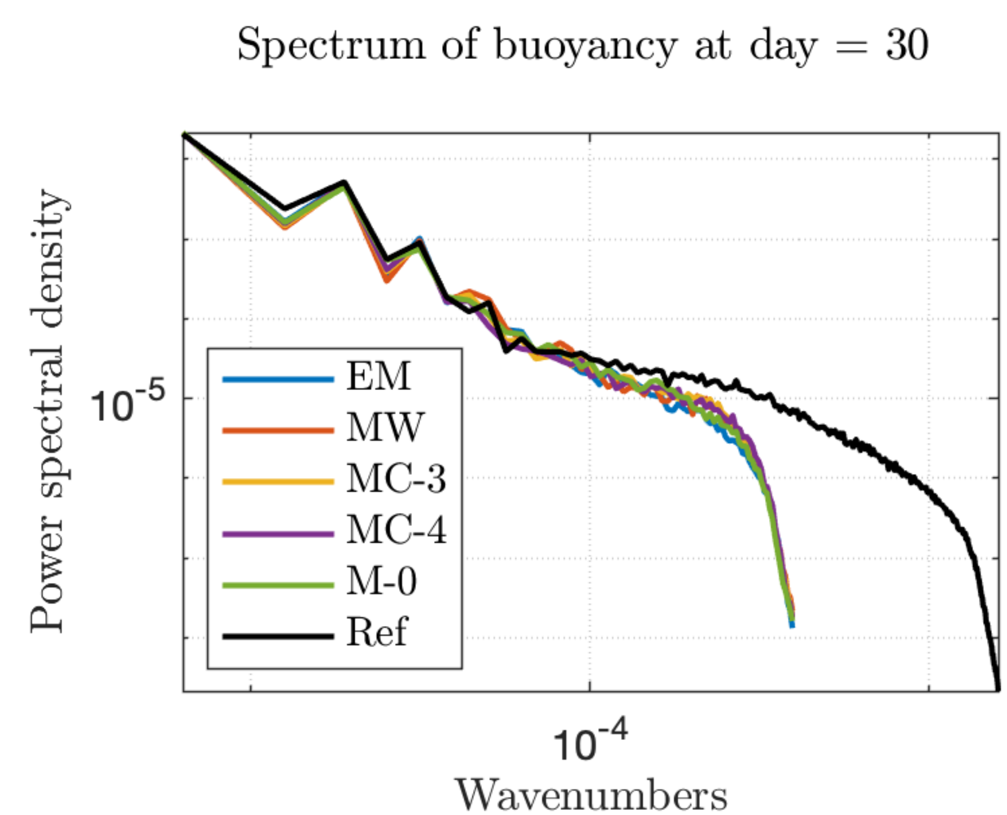
POD noise





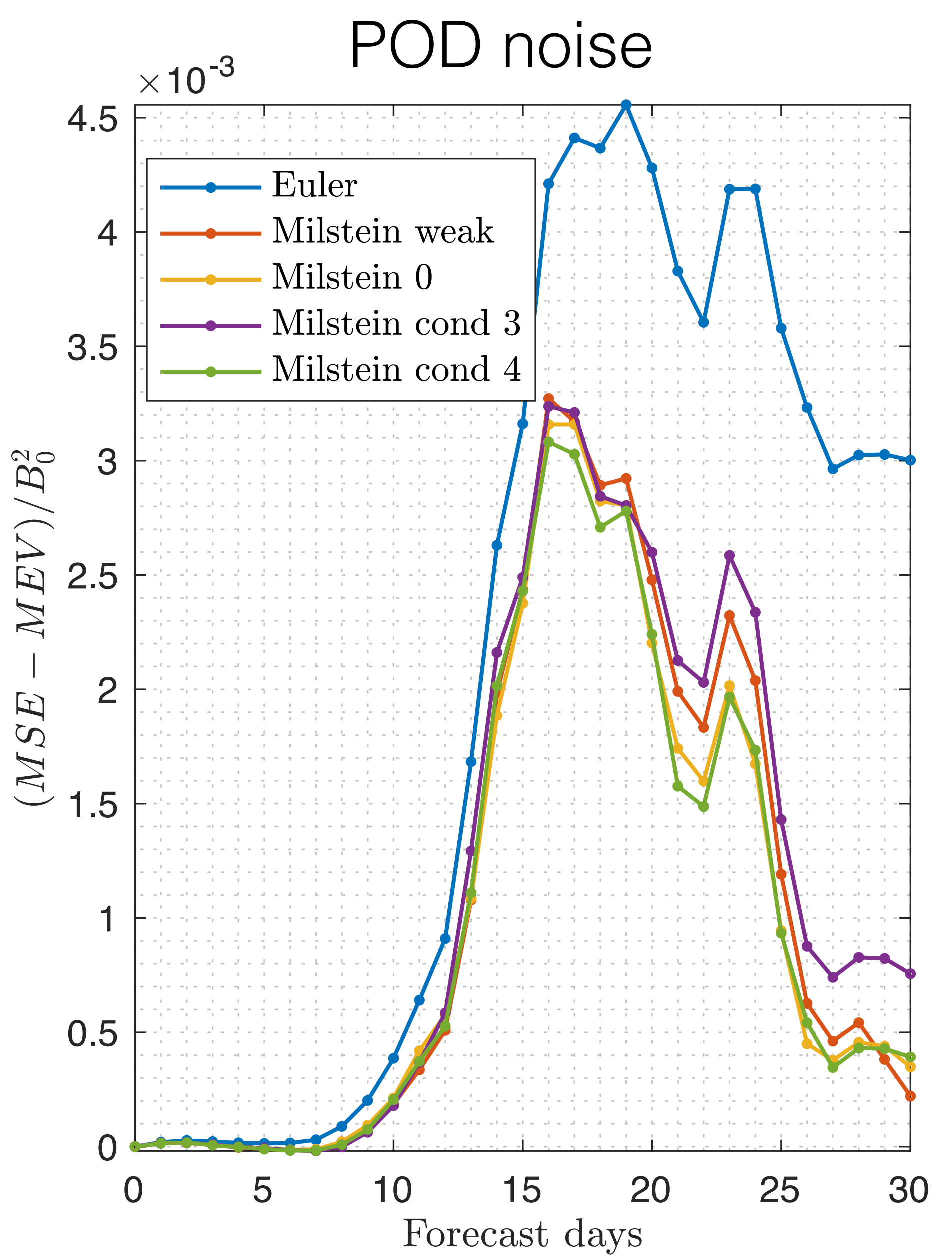
SVD noise

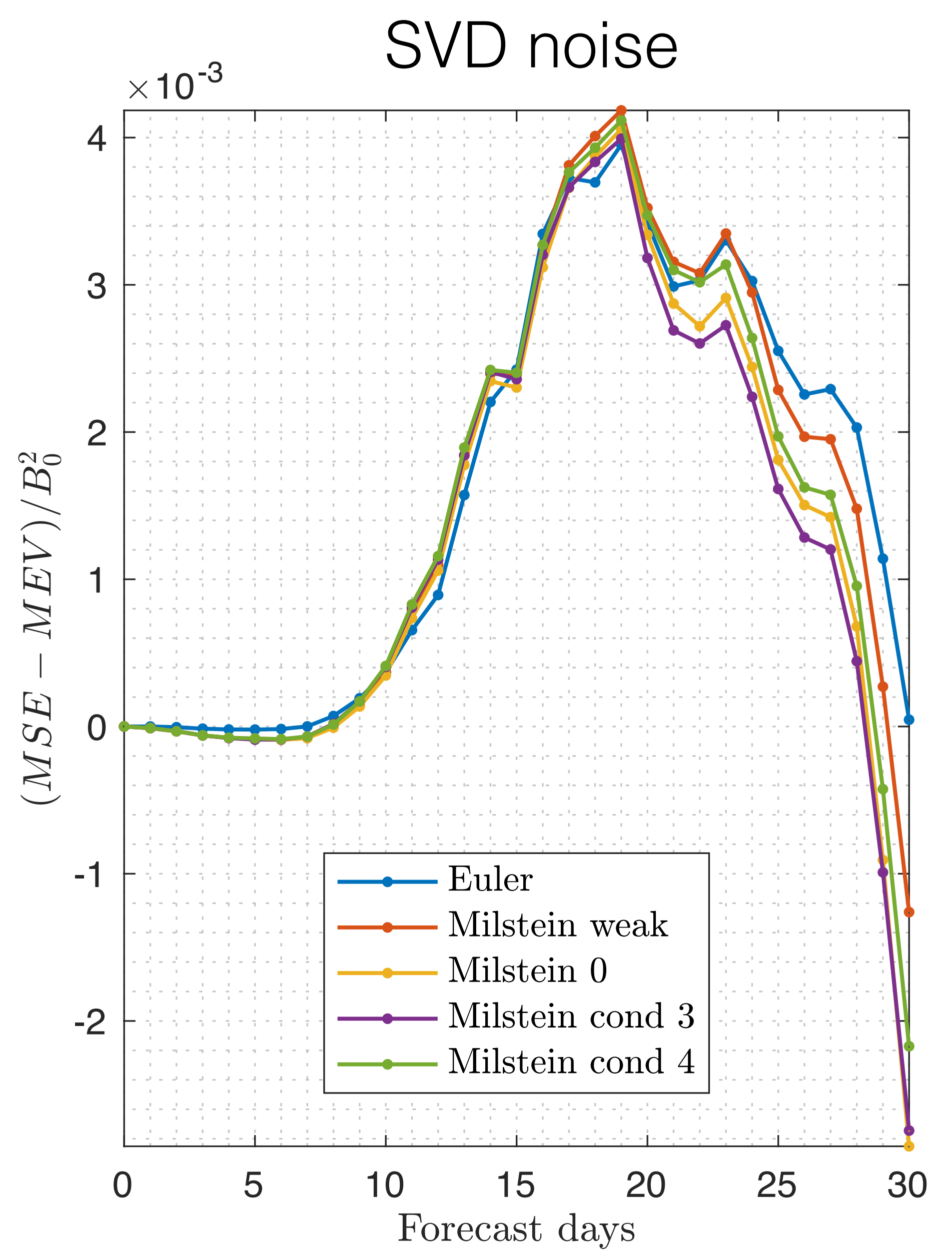




A necessary condition for ensemble reliability is that the mean squared error (MSE) of the ensemble mean forecast is close to the mean ensemble variance:

$$\text{MSE} = \sum_{n=1}^{N} \left(\hat{\mathbb{E}}[b_n] - b_n^{obs} \right)^2 \simeq C \left(\frac{1}{N} \sum_{n=1}^{N} \hat{\text{Var}}[b_n] \right) = \text{MEV}$$

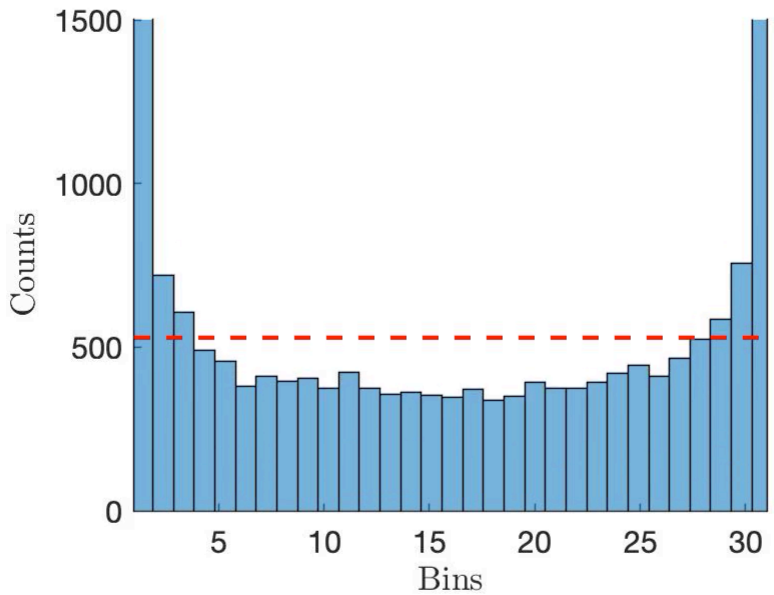




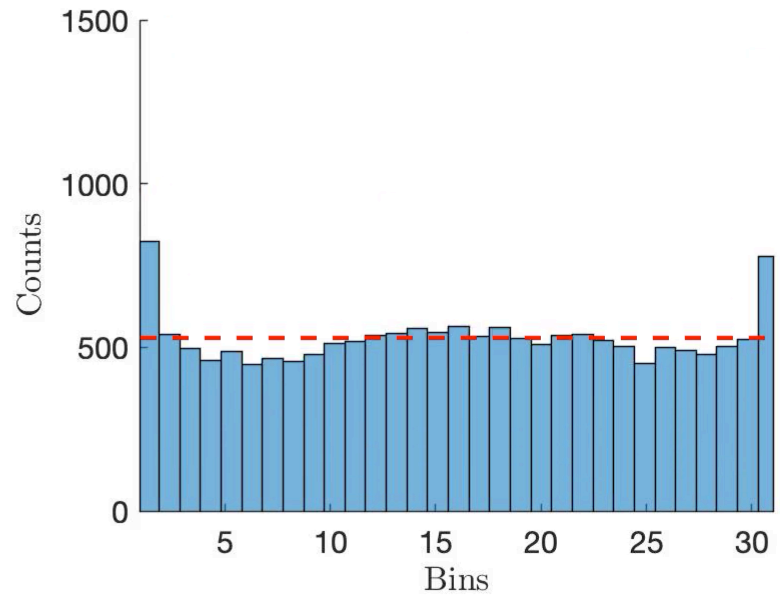
Numerical results

POD noise

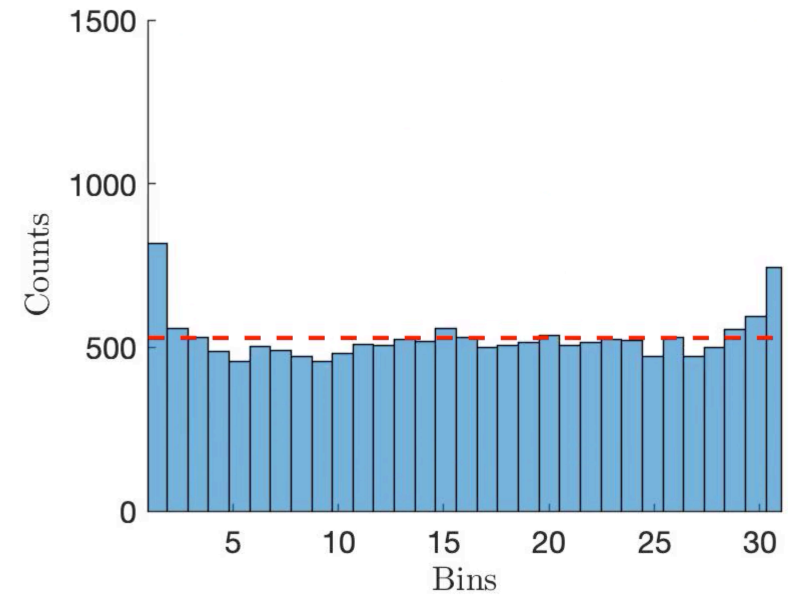
Euler Maruyama



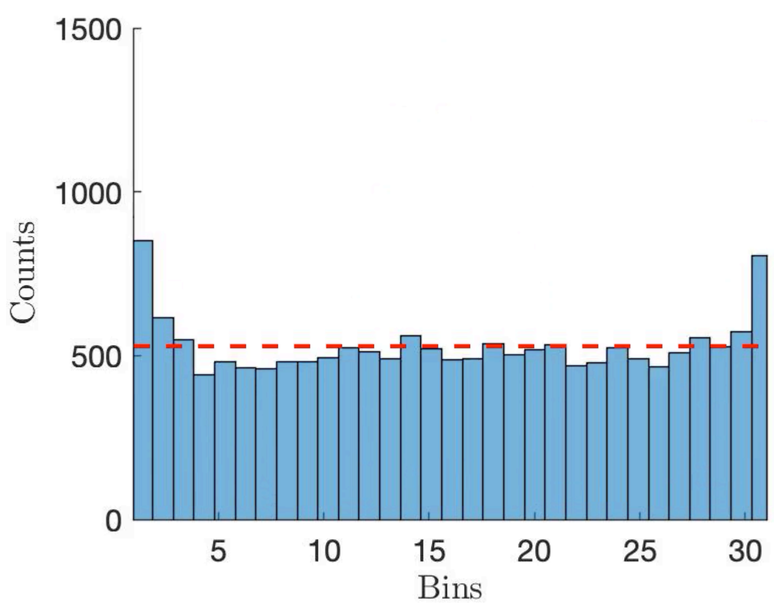
Milstein weak



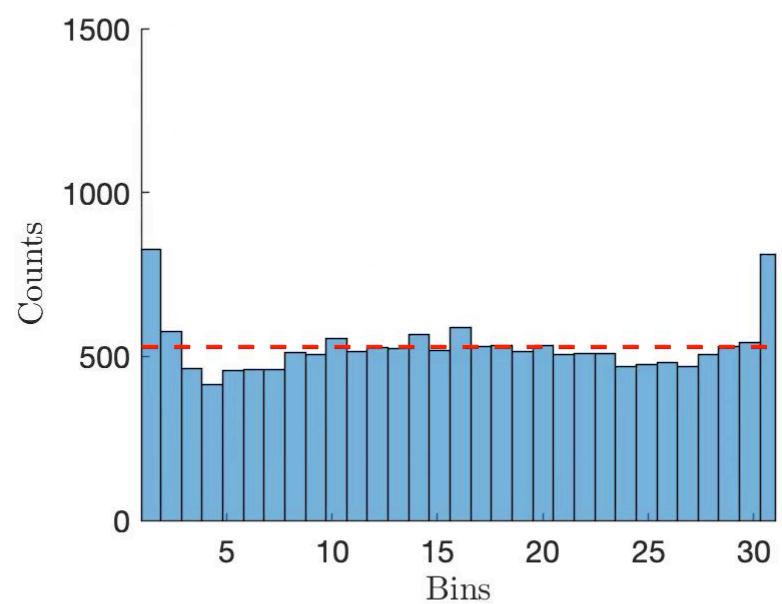
Milstein 0



Conditional 3



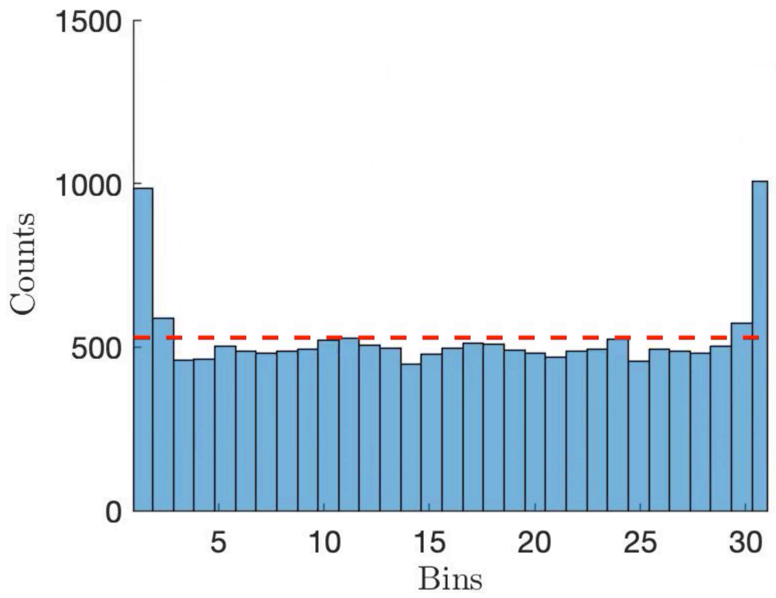
Conditional 4



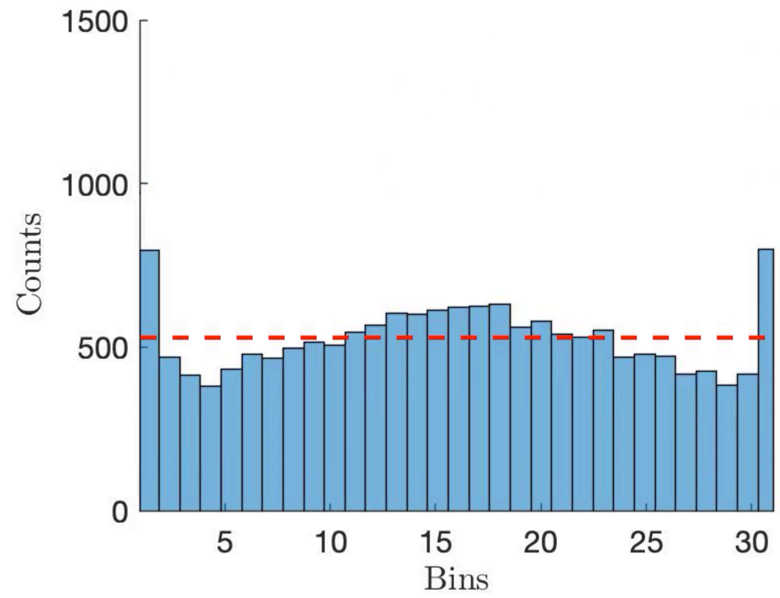
Numerical results

SVD noise

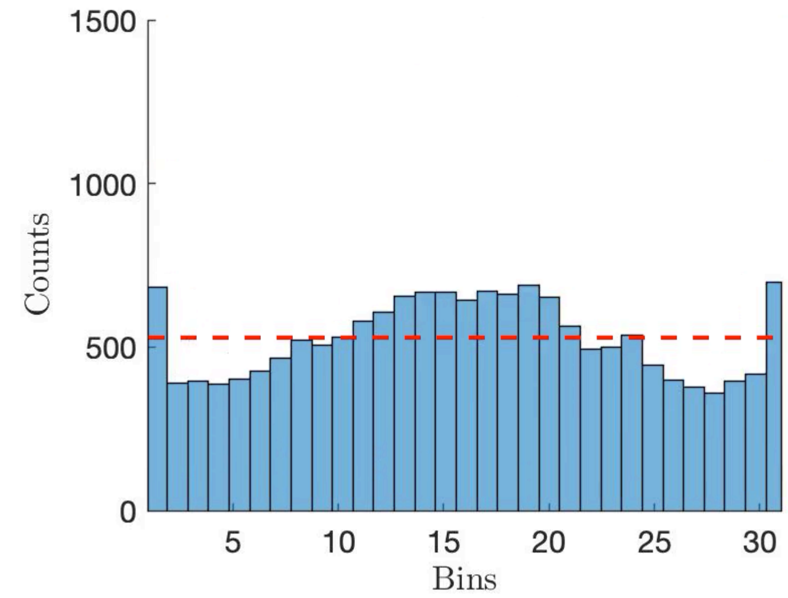
Euler Maruyama



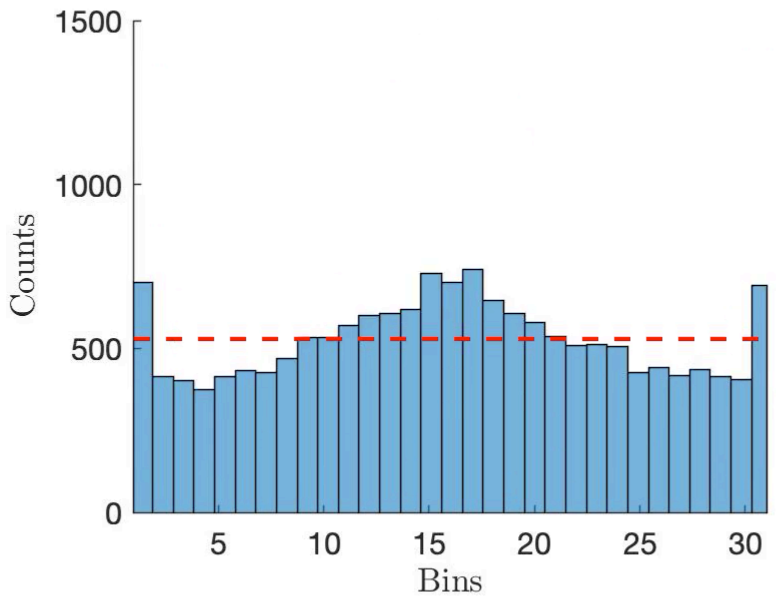
Milstein weak



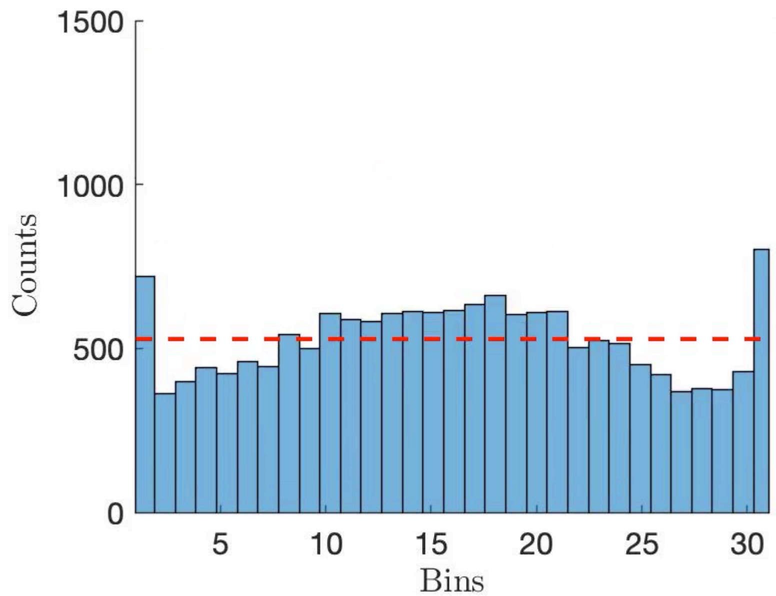
Milstein 0



Conditional 3



Conditional 4



Is $G^{m,k}$ symmetric after all?

We recall the following definition:

$$G^{m,k} = \nabla \cdot \left(\varphi_m \varphi_k^T \nabla b\right)$$

And introduce the following matrices:

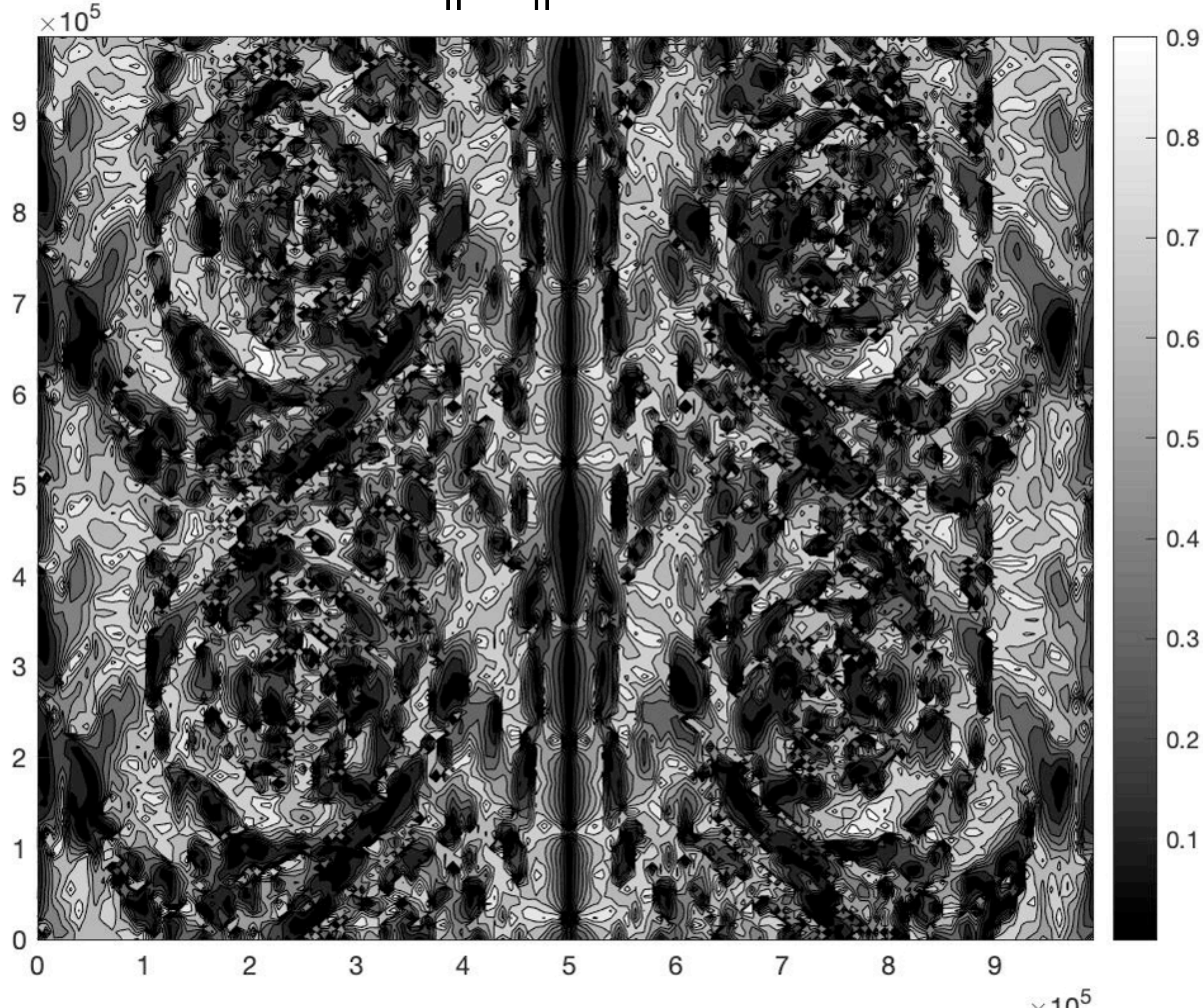
$$\Phi^{m,k} := \varphi_m \varphi_k^T$$

The symmetric and antisymmetric parts are:

$$\Phi^A = \frac{\Phi^{m,k} - \Phi^{k,m}}{2}$$

$$\Phi^S = \frac{\Phi^{m,k} + \Phi^{k,m}}{2}$$

$$\frac{\|\Phi^A\|}{\|\Phi^S\|} \quad \forall m, k$$ for the POD noise



Conclusion and perspectives

Conclusion

- Milstein schemes improve significantly the numerical results;
- The Lévy area does not seem to play a key role in these test cases.

Perspectives

- Understand if the (non) importance of the Lévy area is related to the test case, the equations, or other factors
- Apply this numerical scheme to other equations, starting with barotropic QG.