Sensitivity analysis for hyperbolic PDEs systems with discontinuous solution

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École de recherche en mathématiques pour l'énergie nucléaire

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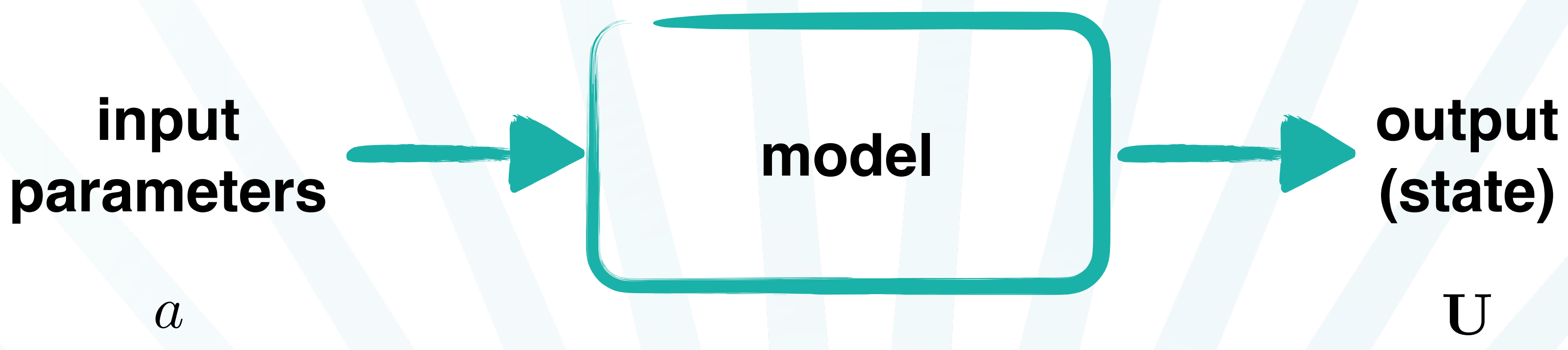
Outline of the talk

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Riemann problem for the Euler equations and their sensitivity
- Classical numerical schemes
- Anti-diffusive numerical schemes
- Numerical results
- Uncertainty quantification

Sensitivity Analysis

Sensitivity analysis: study of how changes in the **inputs** of a model affect the **outputs**



Sensitivity: $$\frac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$$

Standard techniques of sensitivity analysis

Standard techniques of sensitivity analysis call for the differentiation of the state system:

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 & \Omega \times (0, T), \\ \mathbf{U}(x, 0) = \mathbf{g}(x) & \Omega, \end{cases}$$

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This can be done under **hypotheses of regularity** of the state ***U***.

If these techniques are applied to hyperbolic equations, **Dirac delta functions** will appear in the sensitivity.

Sensitivity analysis for hyperbolic equations

Definition of the source term

In order to have a sensitivity system which is valid also when the state is discontinuous, we add a correction term:

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S} & \Omega \times (0, T), \\ \mathbf{U}_a(x, 0) = \mathbf{g}_a(x) & \Omega, \end{cases}$$

defined as follows:

$$\mathbf{S} = \sum_{k=1}^{N_s} \boldsymbol{\rho}_k \delta(x - x_{k,s}(t)),$$

N_s: number of discontinuities

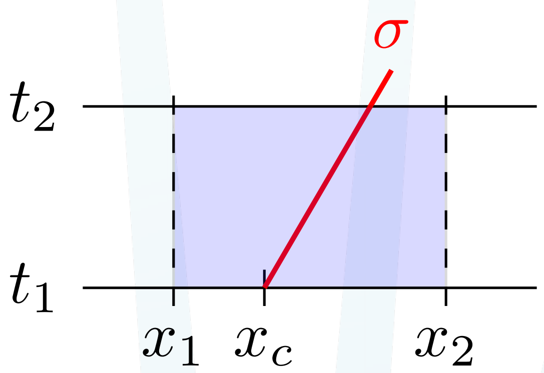
$x_{k,s}$: position of the k-th discontinuity

$\boldsymbol{\rho}_k$: amplitude of the k-th correction (to be computed)

Remark: a **shock detector** is necessary to discretise such source term.

Definition of the source term

To compute the amplitude of the correction, we consider an infinitesimal control volume containing a single discontinuity:



By integrating the sensitivity equations with the source term on the control volume, one has:

$$\boldsymbol{\rho}_k = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + \mathbf{F}_a^+ - \mathbf{F}_a^-$$

Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma_k = \mathbf{F}^+ - \mathbf{F}^-$

Differentiating them w.r.t. the parameter:

$$(\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + (\mathbf{U}^- - \mathbf{U}^+)\sigma_{k,a} + \sigma_k(\partial_x \mathbf{U}^+ - \partial_x \mathbf{U}^-)\partial_a x_{k,s}(t) =$$
$$= \mathbf{F}_a^- - \mathbf{F}_a^+ + \left(\frac{\partial \mathbf{F}(\mathbf{U}^+)}{\partial \mathbf{U}}\partial_x \mathbf{U}^+ - \frac{\partial \mathbf{F}(\mathbf{U}^-)}{\partial \mathbf{U}}\partial_x \mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

Finally, we obtain the following amplitude: $\boldsymbol{\rho}_k = (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{a,k}$

Riemann problem for Euler equations and their sensitivity

The Riemann problem for Euler equations

The Euler equations write:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = 0, \end{cases}$$

Eigenvalues:	Eigenvectors:
$\lambda_1(\mathbf{U}) = u - c,$	$\mathbf{r}_1(\mathbf{U}) = (1, u - c, H - uc)^t,$
$\lambda_2(\mathbf{U}) = u,$	$\mathbf{r}_2(\mathbf{U}) = (1, u, \frac{u^2}{2})^t,$
$\lambda_3(\mathbf{U}) = u + c.$	$\mathbf{r}_3(\mathbf{U}) = (1, u + c, H + uc)^t.$

Genuinely nonlinear

The Riemann problem for Euler equations

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Linearly degenerate

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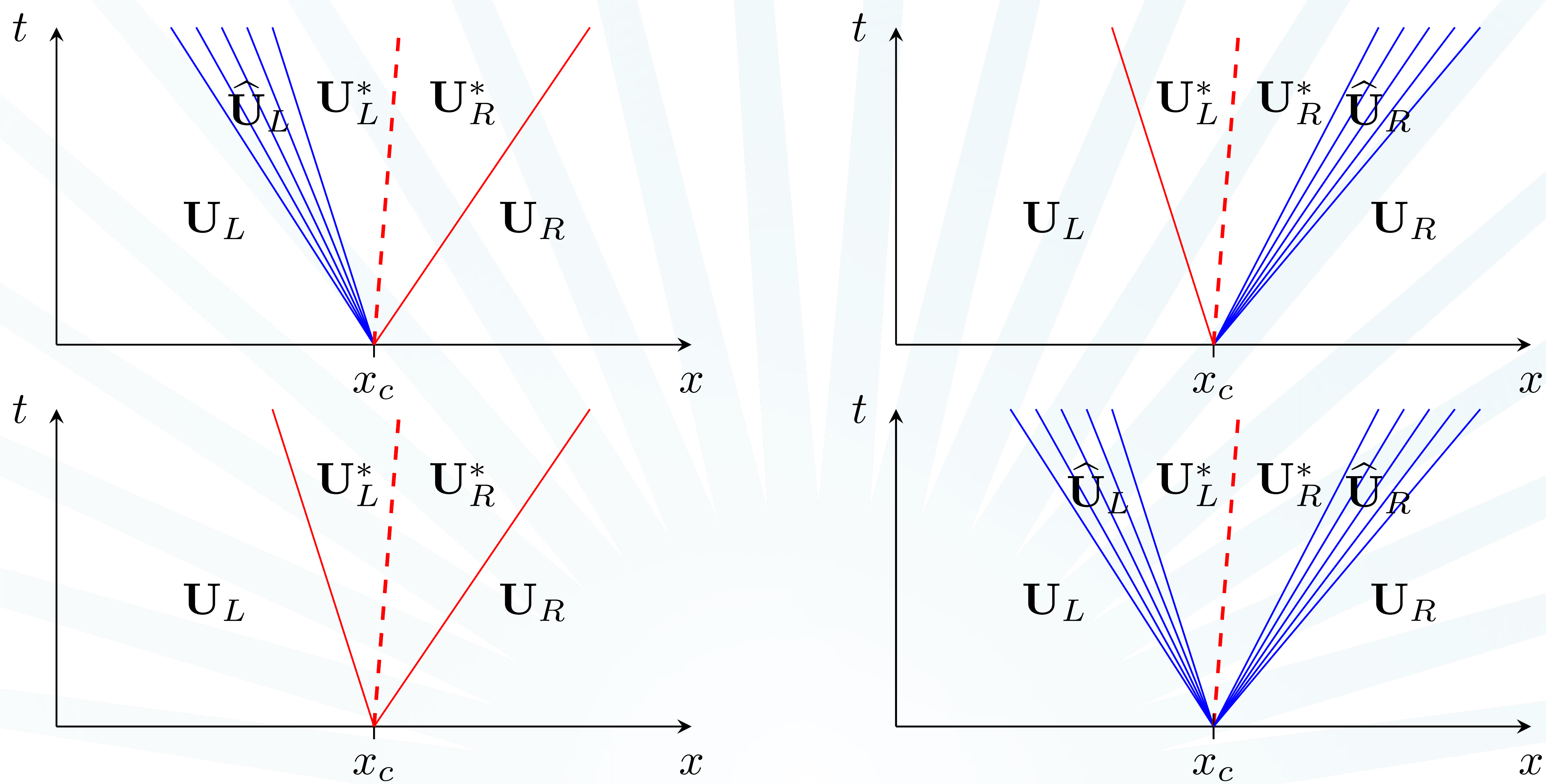


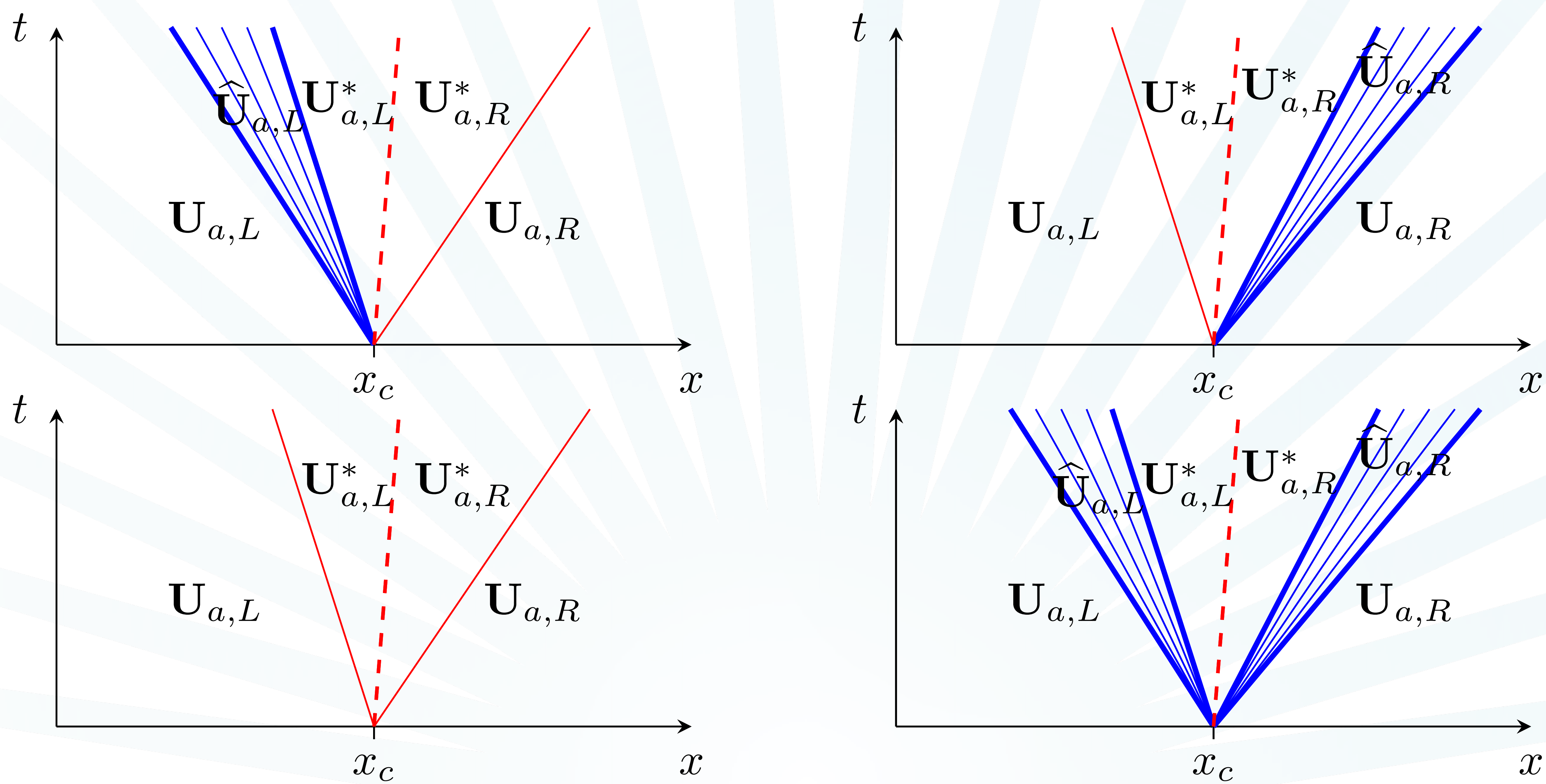
The Riemann problem for the sensitivity equations

The sensitivity system writes:

$$\begin{cases} \partial_t \rho_a + \partial_x (\rho u)_a = S_1, \\ \partial_t (\rho u)_a + \partial_x (\rho_a u^2 + 2\rho u u_a + p_a) = S_2, \\ \partial_t (\rho E)_a + \partial_x (u_a(\rho E + p) + u((\rho E)_a + p_a)) = S_3, \end{cases}$$

Eigenvalues:

$$\lambda_1(\mathbf{U}) = u - c,$$
$$\lambda_2(\mathbf{U}) = u,$$
$$\lambda_3(\mathbf{U}) = u + c.$$



Classical Numerical Schemes

Godunov-type schemes

approximate Riemann solvers are used

Step 0 : initial data discretisation

Step 1 : solution of a Riemann problem for each interface $x_{j-1/2}$ obtaining $\mathbf{v}(x, t^{n+1})$

Step 2 : average $$\mathbf{V}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, t^{n+1}) \mathrm{d}x$$

Remark: the state and the sensitivity systems are solved **separately**.

Classical numerical schemes

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

Remark: HLL-type schemes cannot be used for the state, two intermediate star states are necessary to have a well-defined the source term for the sensitivity.

Approximate Riemann solver for the state

First order Roe-type scheme

$\lambda_1^{ROE} = \tilde{u} - \tilde{c} \qquad \lambda_2^{ROE} = \tilde{u} \qquad \lambda_3^{ROE} = \tilde{u} + \tilde{c}$ Roe-averaged eigenvalues

$\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \alpha_i \tilde{\mathbf{r}}_i$ decomposition along Roe-averaged eigenvectors

$$\mathbf{U}_L^* = \mathbf{U}_L + \alpha_1 \tilde{\mathbf{r}}_1 \qquad \mathbf{U}_R^* = \mathbf{U}_R - \alpha_3 \tilde{\mathbf{r}}_3$$

Approximate Riemann solvers for the sensitivity

- HLL-type scheme: simpler structure that the state solver.

HLL consistency conditions yield:

$$\mathbf{U}^*_{a,j-1/2} = \frac{1}{\lambda_3^{ROE} - \lambda_1^{ROE}} \Big(\lambda_3^{ROE} \mathbf{U}^n_{a,j} - \lambda_1^{ROE} \mathbf{U}^n_{a,j-1} - \mathbf{F}_a(\mathbf{U}_j, \mathbf{U}_{a,j}) + \mathbf{F}_a(\mathbf{U}_{j-1}, \mathbf{U}_{a,j-1}) + \mathbf{S}_{j-1/2} \Big)$$

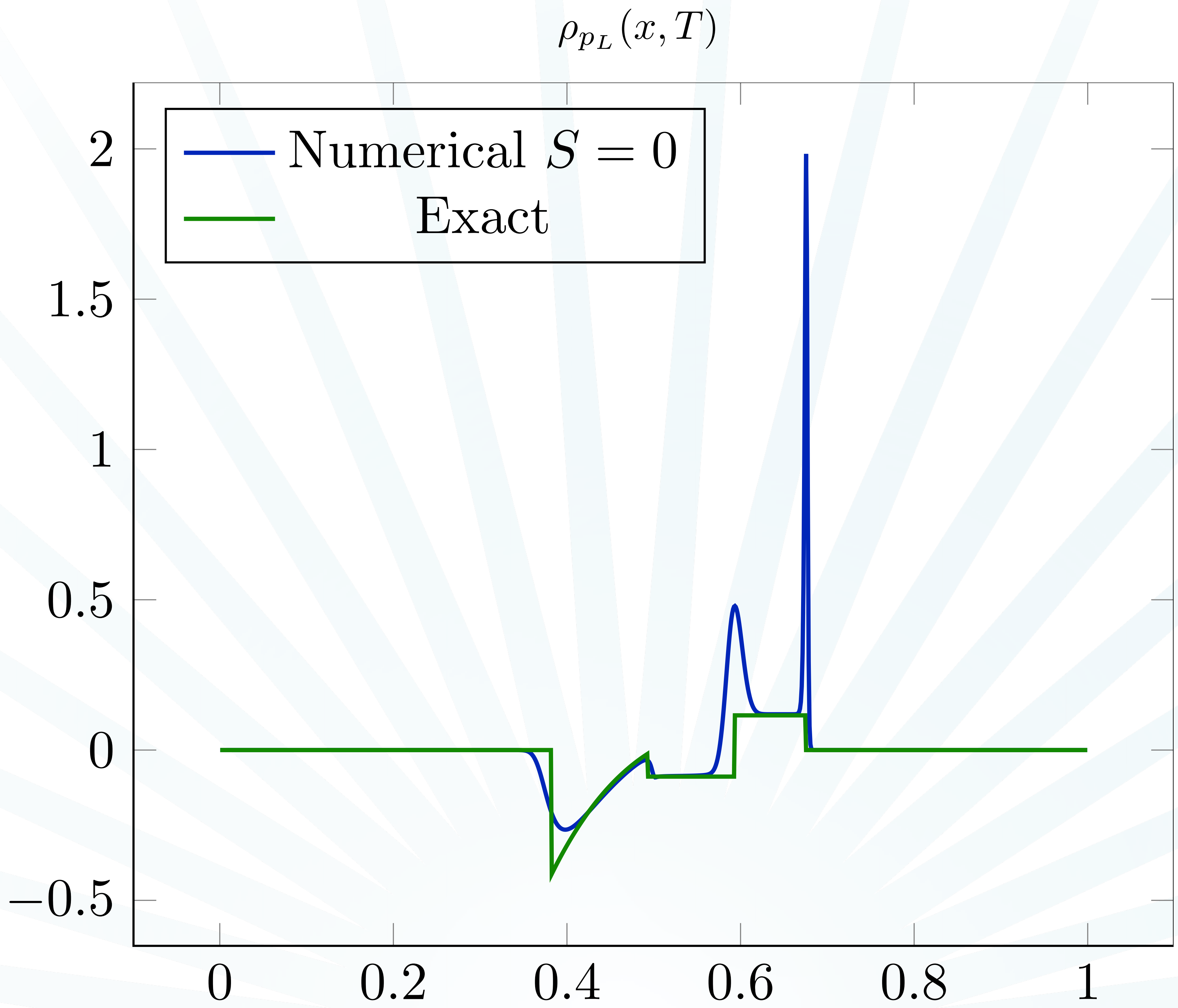
$$\begin{aligned} \mathbf{S}_{j-1/2} =& \partial_a \lambda^{ROE}_{1,j-1/2} (\mathbf{U}^*_{L,j-1/2} - \mathbf{U}_{j-1}) d_{1,j-1/2} \\ &+ \partial_a \lambda^{ROE}_{2,j-1/2} (\mathbf{U}^*_{R,j-1/2} - \mathbf{U}^*_{L,j-1/2}) \\ &+ \partial_a \lambda^{ROE}_{3,j-1/2} (\mathbf{U}_j - \mathbf{U}^*_{R,j-1/2}) d_{3,j-1/2} \end{aligned}$$

- HLLC-type scheme: same structure as the state.

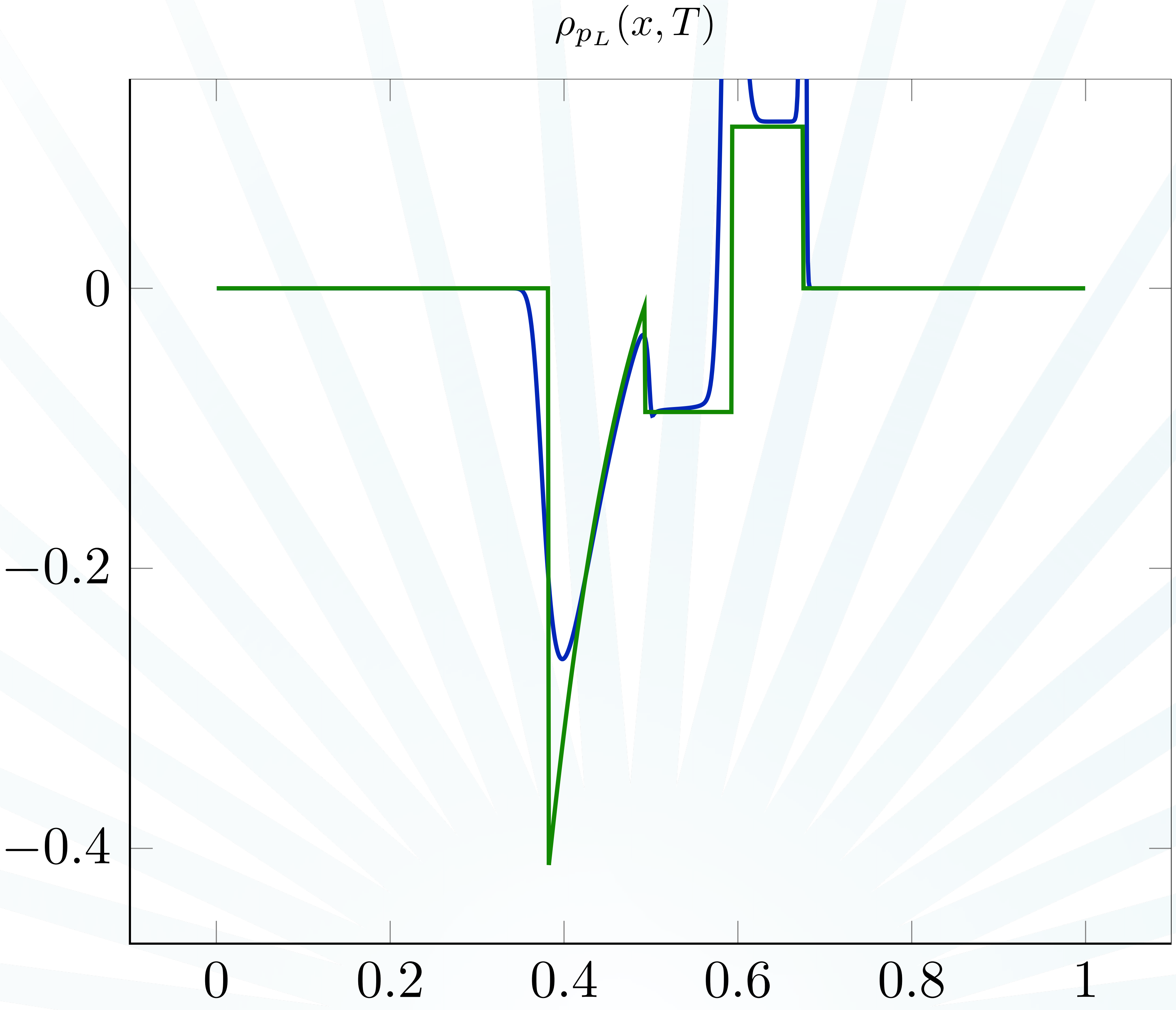
HLL consistency conditions + Rankine-Hugoniot conditions. Equivalent to:

$$\mathbf{U}^*_{a,L} = \mathbf{U}_{a,L} + \alpha_{1,a} \tilde{\mathbf{r}}_1 + \alpha_1 \tilde{\mathbf{r}}_{1,a} \qquad \mathbf{U}^*_{a,R} = \mathbf{U}_{a,R} - \alpha_{3,a} \tilde{\mathbf{r}}_3 - \alpha_3 \tilde{\mathbf{r}}_{3,a}$$

Classical numerical schemes



Classical numerical schemes



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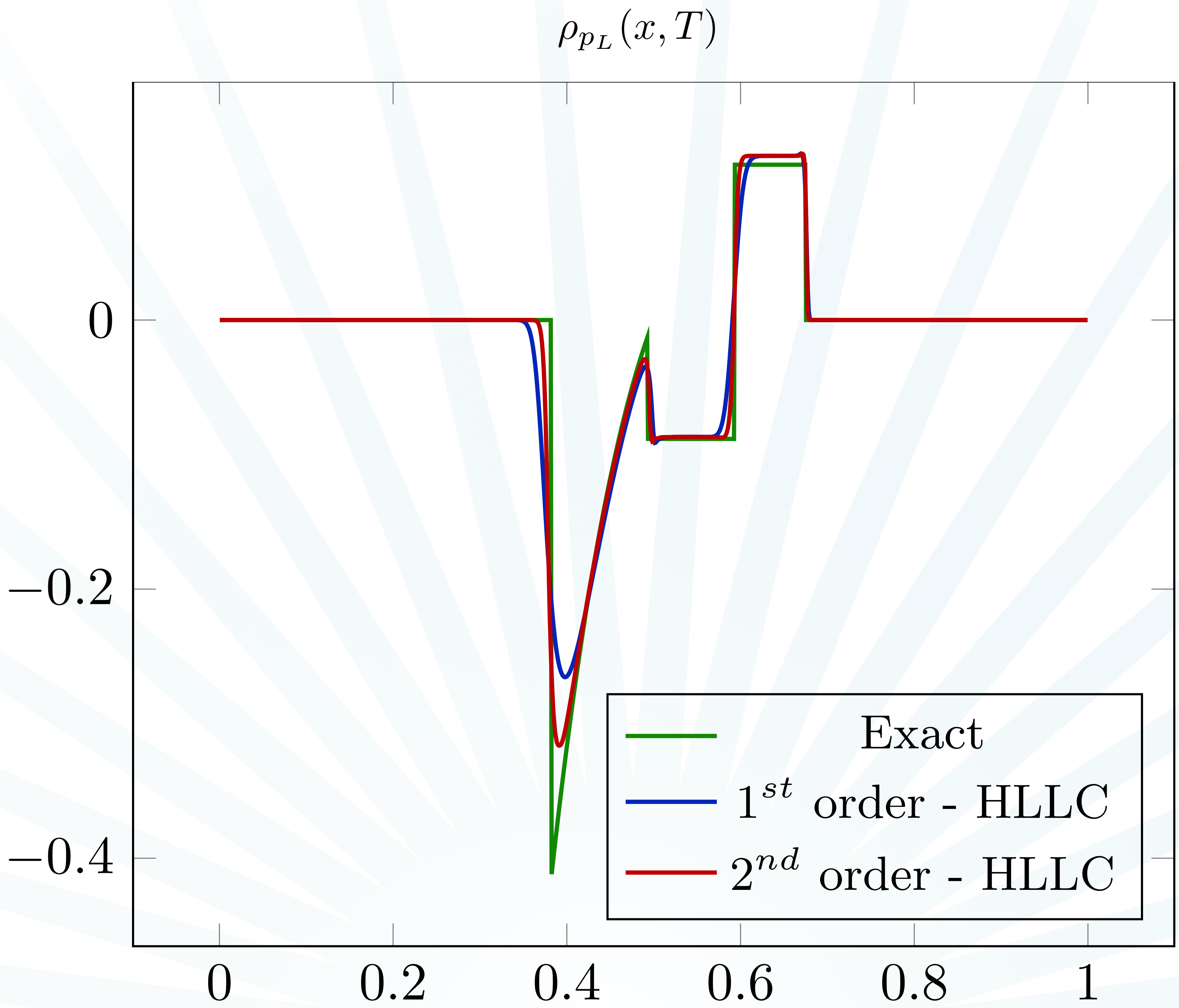


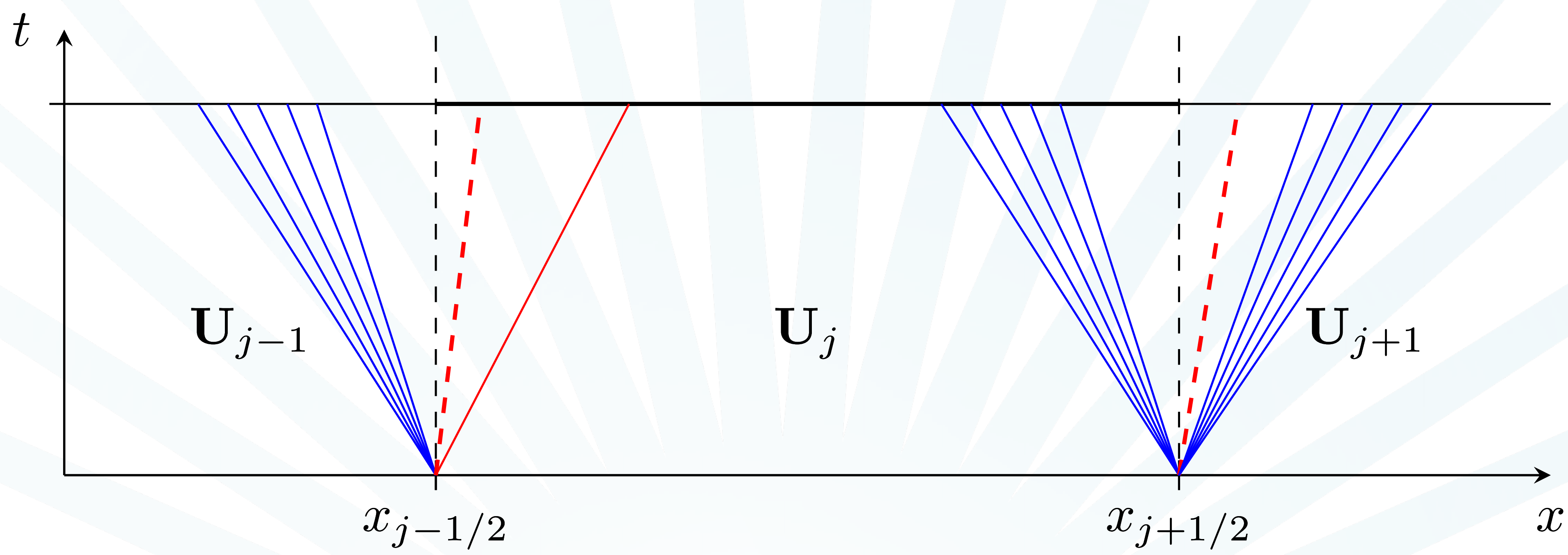
Numerical Scheme without diffusion

Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : ~~average~~

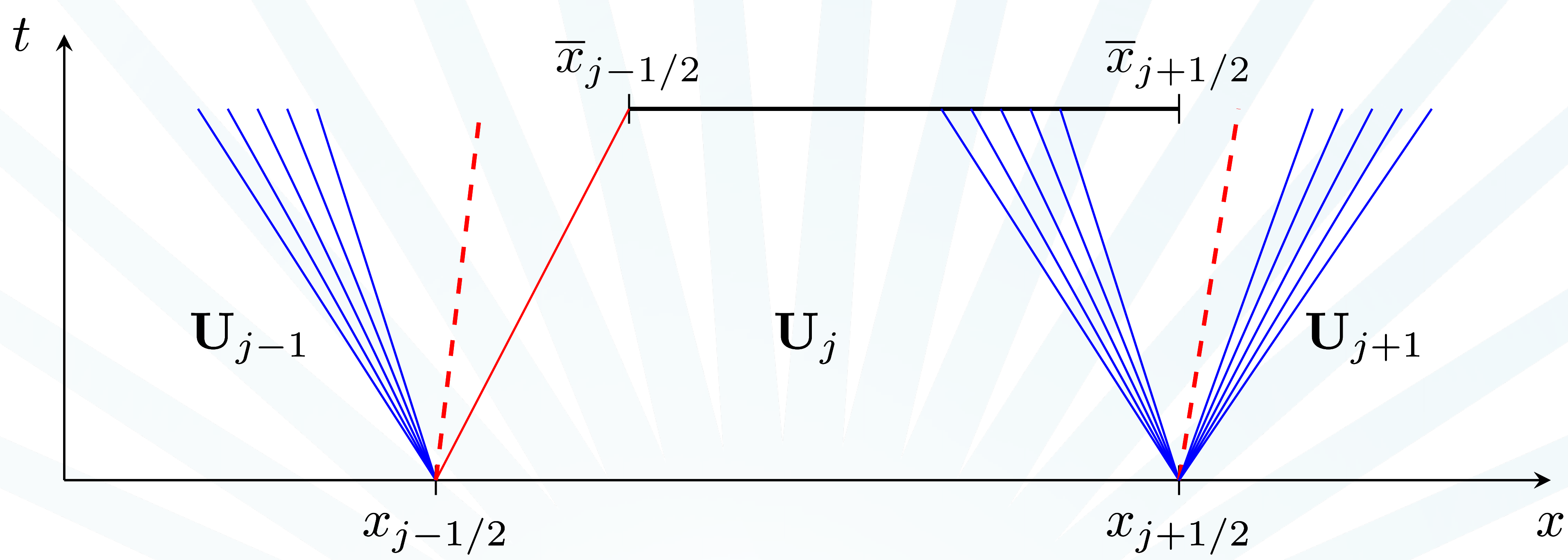


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed



$$\overline{x}_{j-1/2} = x_{j-1/2} + \sigma_{j-1/2}\Delta t$$

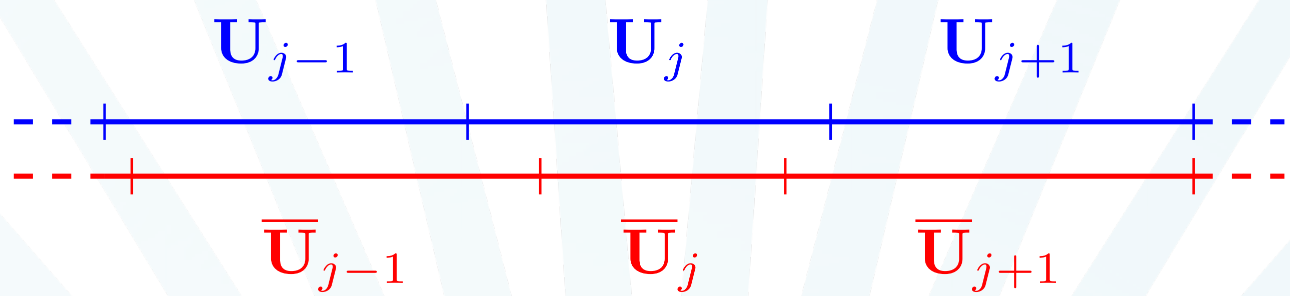
Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed

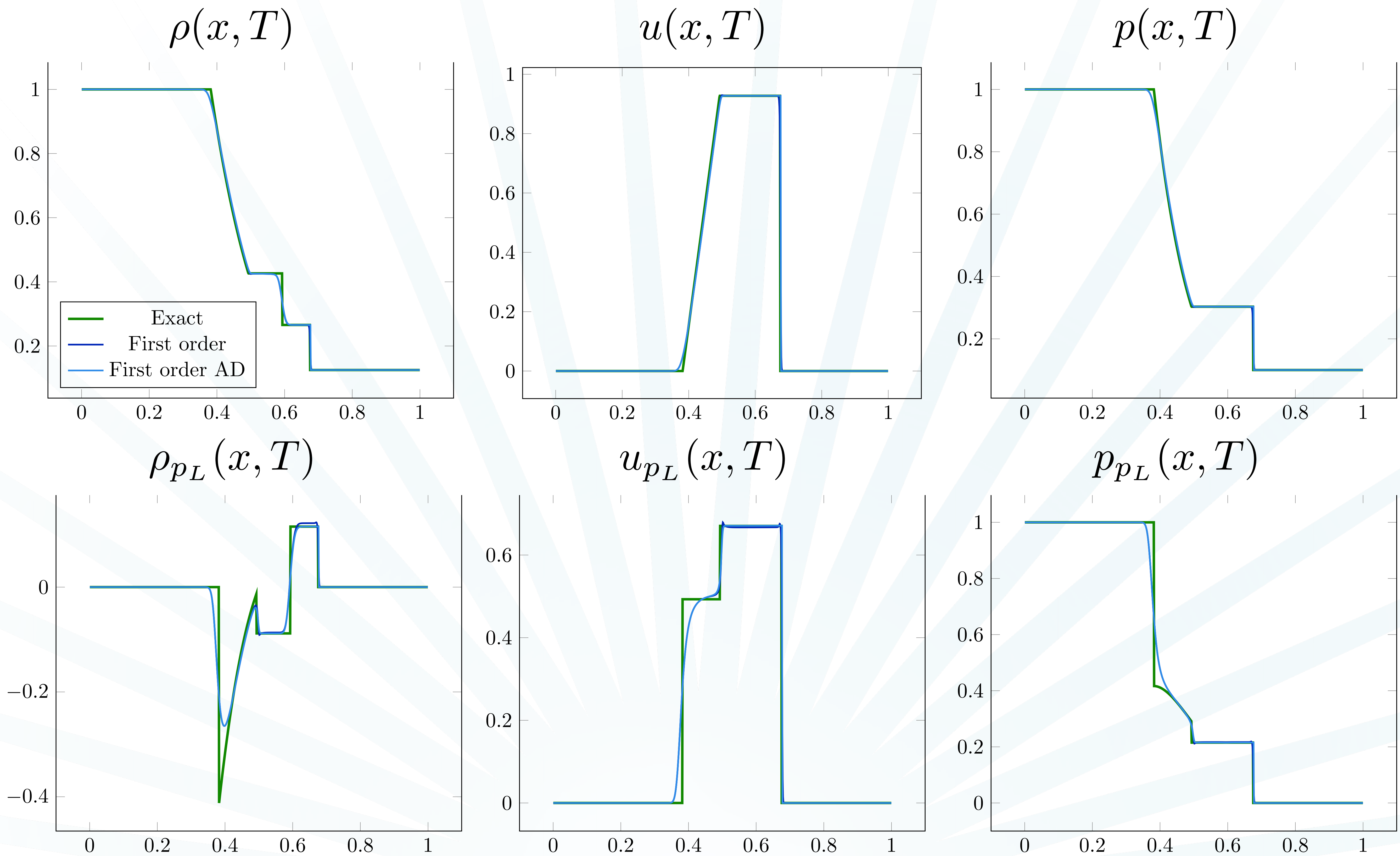
Step 3 : projection on the initial mesh



$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } \alpha \in \left(0, \frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j} & \text{if } \alpha \in \left[\frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } \alpha \in \left[1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0), 1\right). \end{cases}$$

$$\alpha \sim \mathcal{U}([0,1])$$

Numerical results



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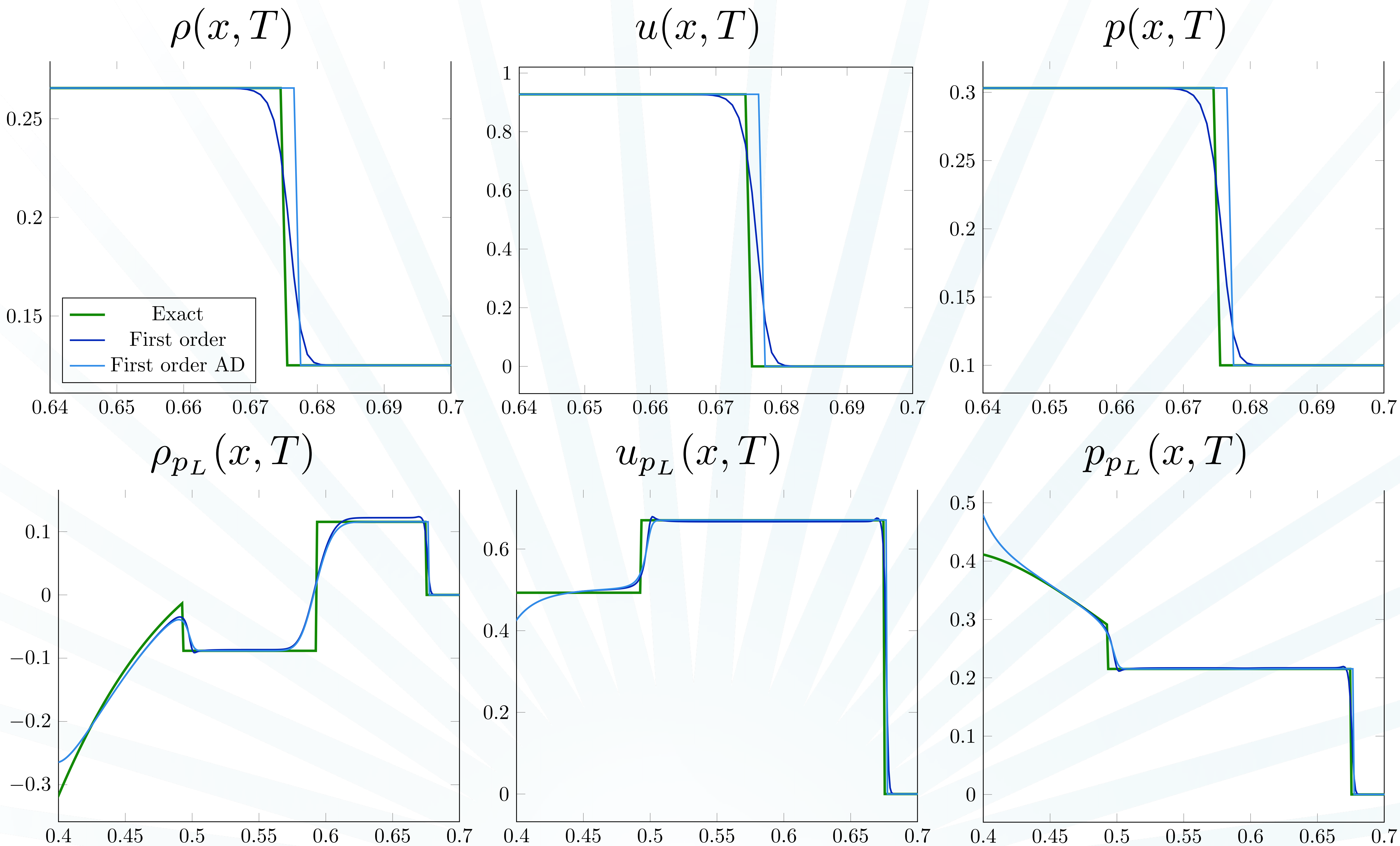
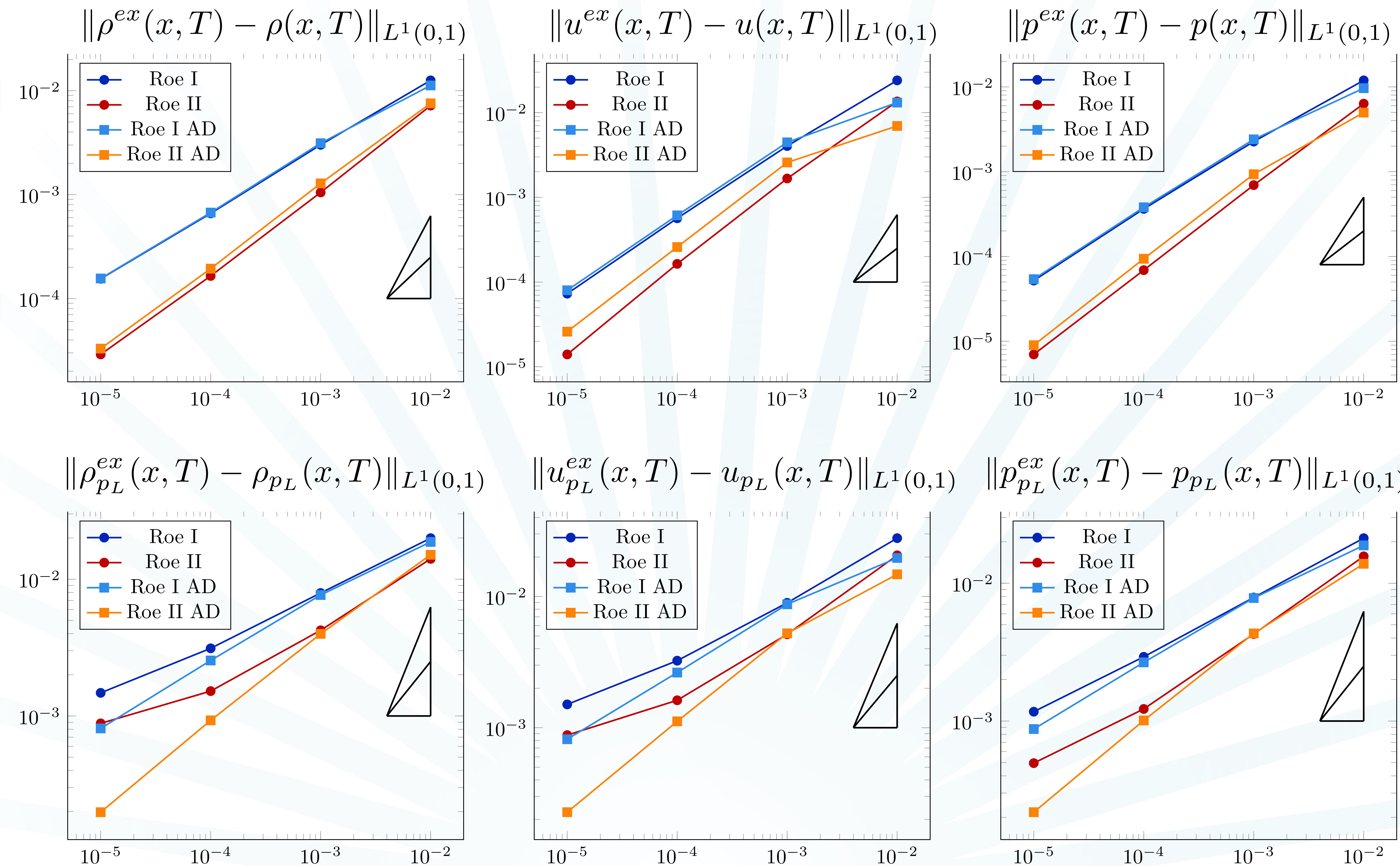


Convergence



Uncertainty quantification

Uncertainty Quantification

Riemann problem with uncertain parameters: $\mathbf{a} = (\rho_L, \rho_R, u_L, u_R, p_L, p_R)^t$

Aim: determine a **confidence interval** $CI_X = [\mu_X - \kappa\sigma_X, \mu_X + \kappa\sigma_X]$

Monte Carlo approach: *N* samples X_k

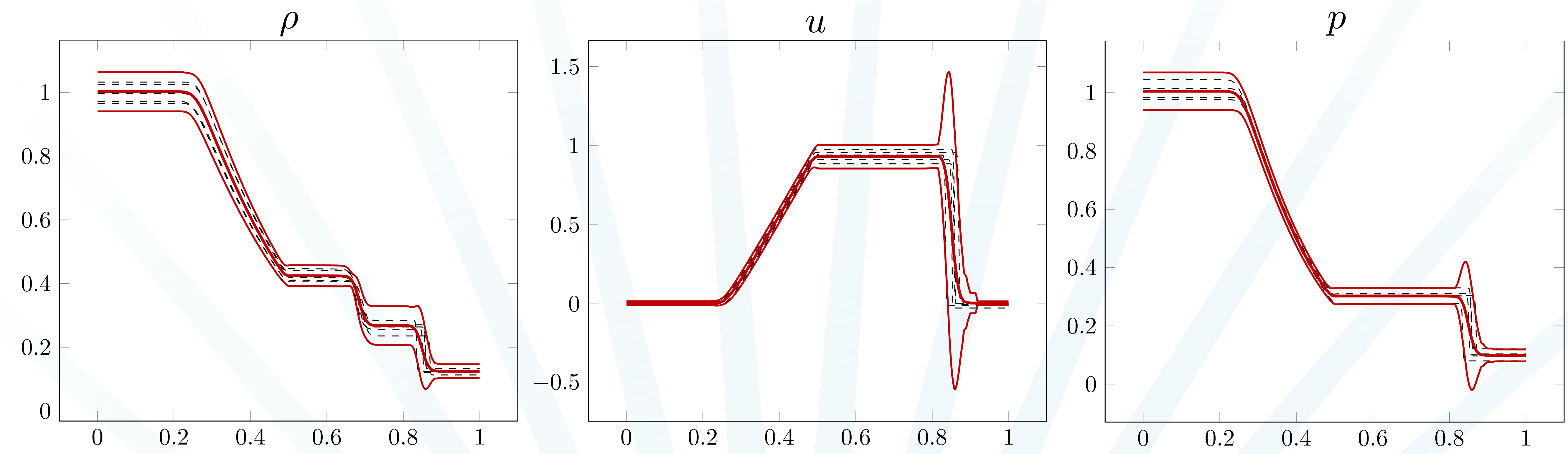
$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma_X^2 = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

Sensitivity approach: state X, sensitivities X_{a_i}

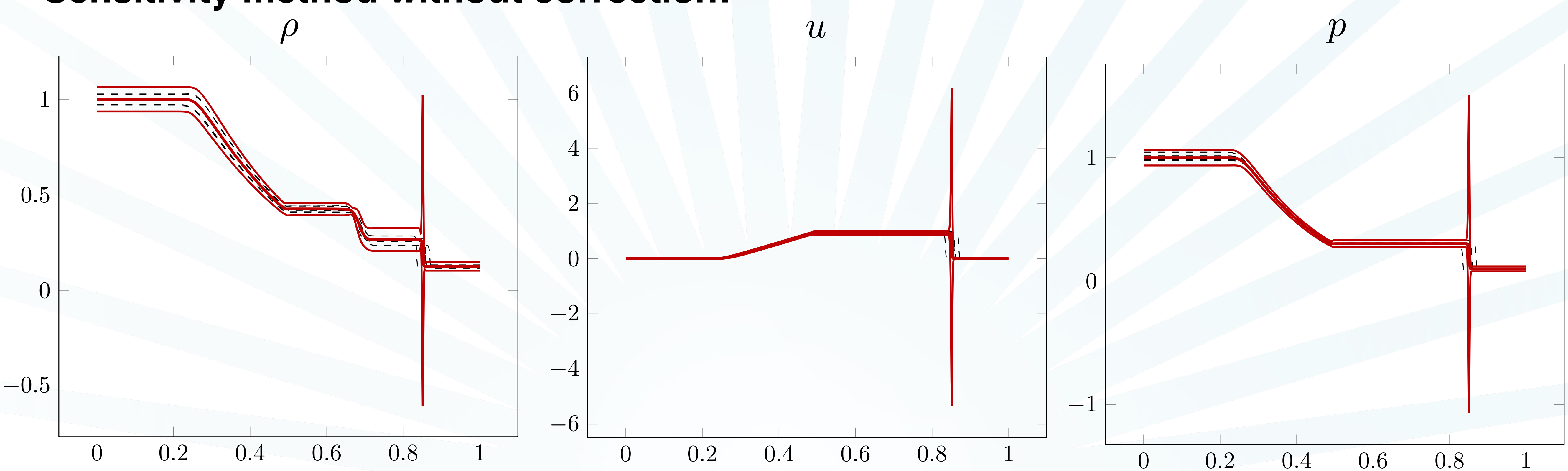
$$\mu_X = X \qquad \sigma_X^2 = \sum_{i=1}^{6} X_{a_i}^2 \sigma_{a_i}^2$$

Uncertainty Quantification

Monte Carlo method:



Sensitivity method without correction:



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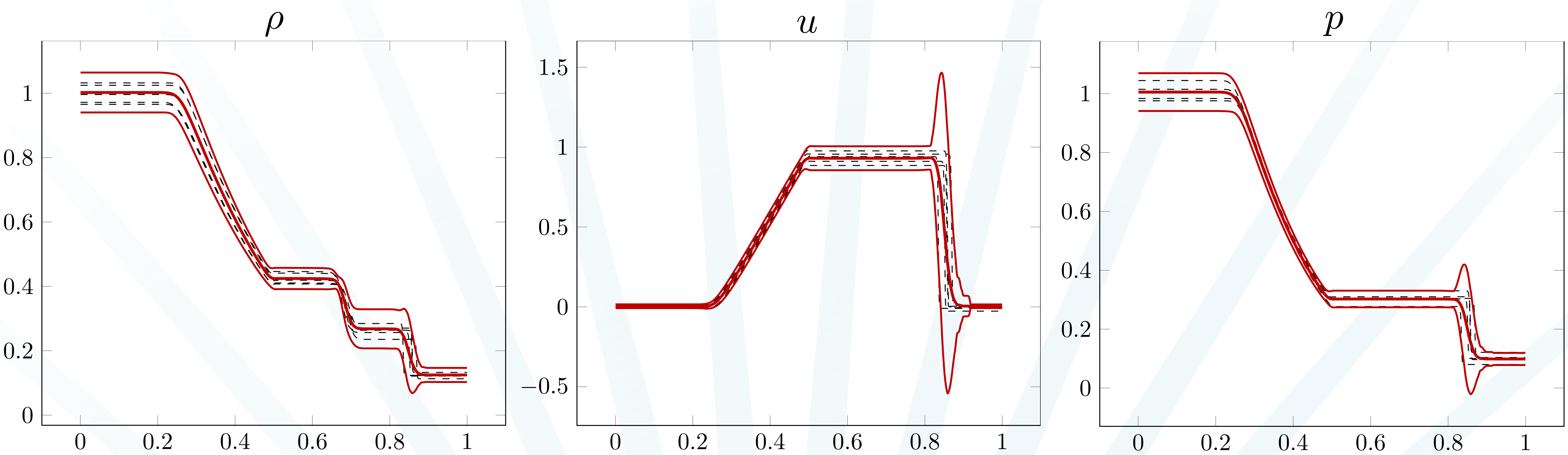

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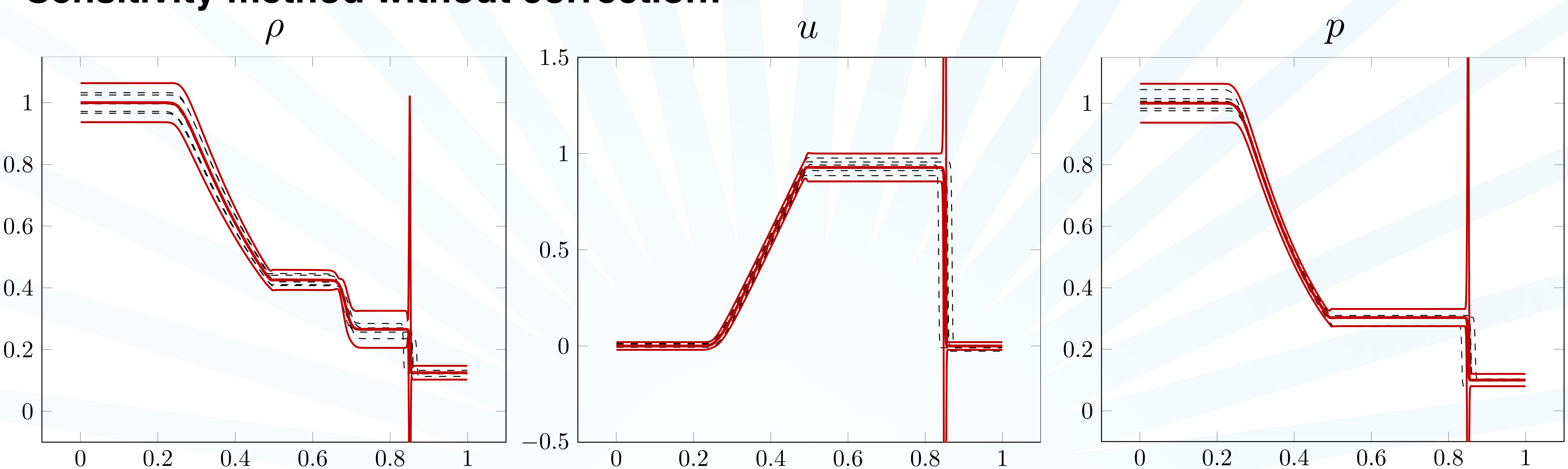
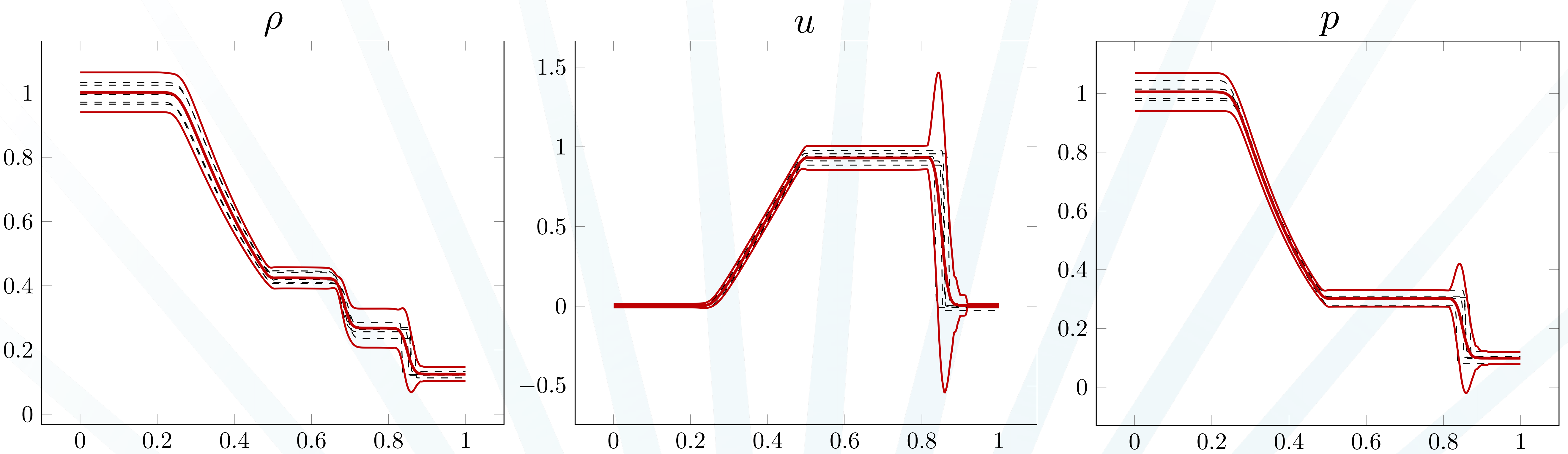
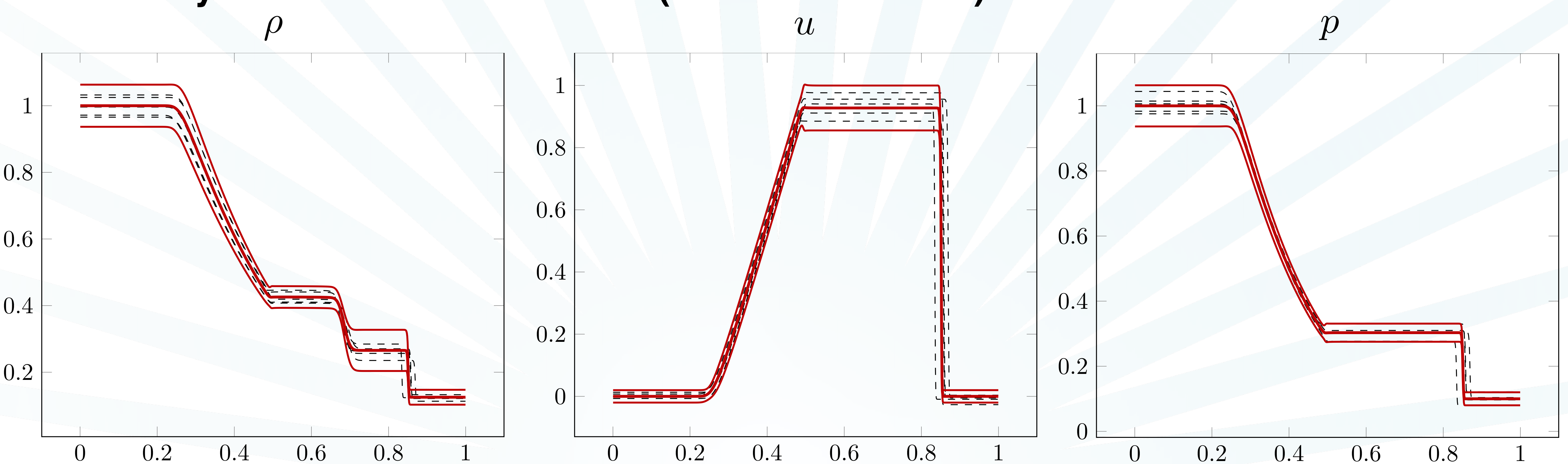


Uncertainty Quantification

Monte Carlo method:



Sensitivity method with correction (diffusive method):



Conclusion and future development

Conclusion:

- We defined a sensitivity system valid in case of discontinuous state
- The correction term is well defined
- The correction term is important in applications

Future development:

- Effects of the numerical diffusion for the applications
- Extension to 2D

Thank you

for your attention!