# Uncertainty quantification for the Navier—Stokes equations

Tiger SHARK-FV

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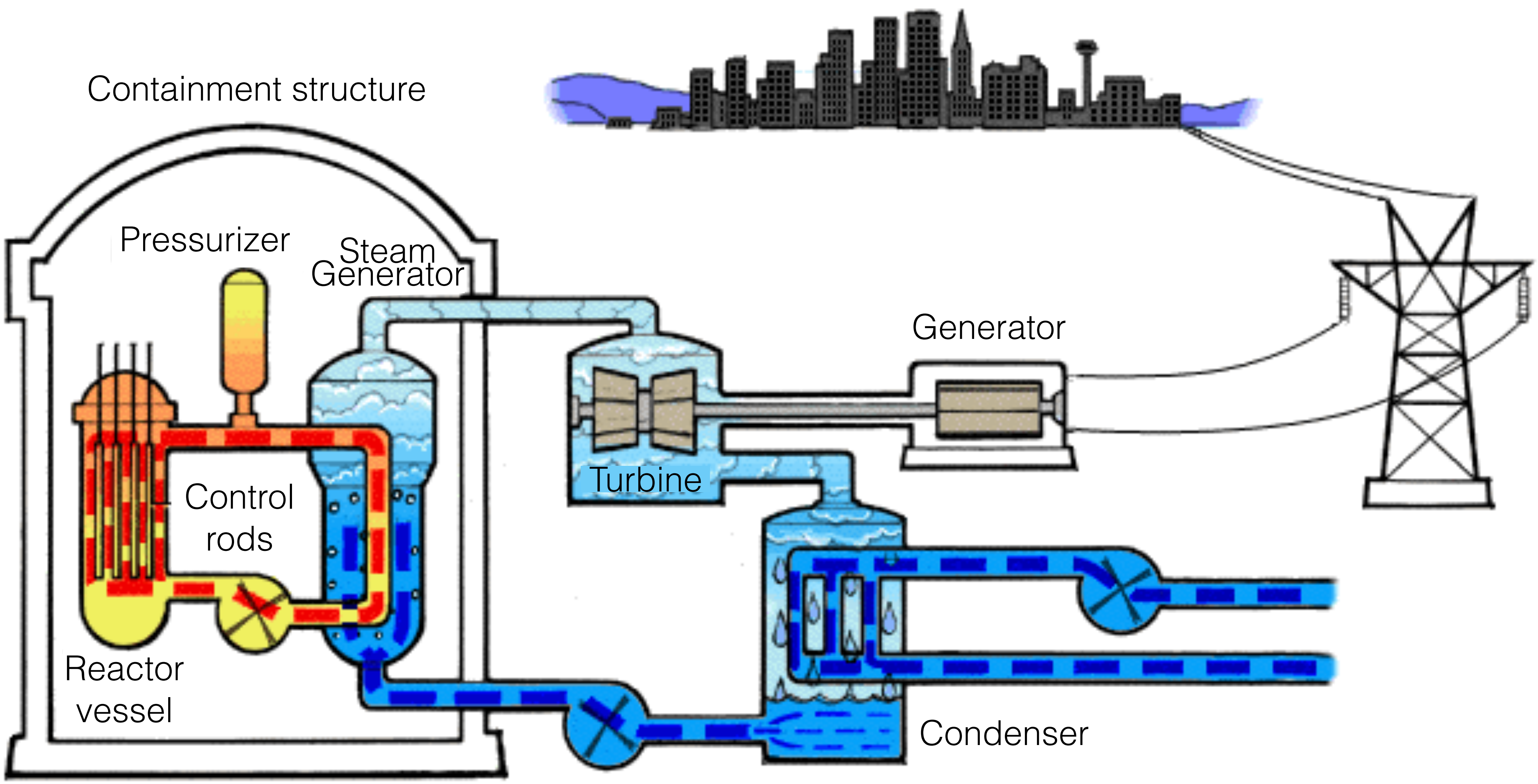
# Context


Containment structure
Pressurizer
Steam
Generator
Control
rods
Reactor
vessel
Turbine
Generator
Condenser

# Context

**Code TRUST TrioCFD:**

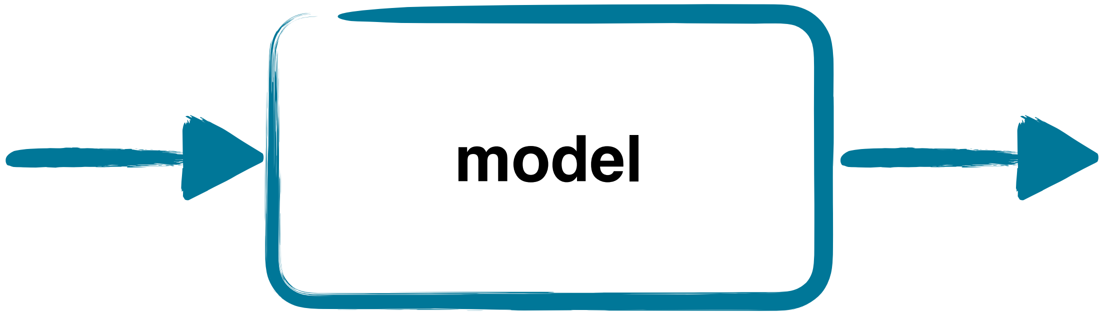
- Developed at CEA
- Open source
- Solves 2D and 3D Navier-Stokes
- **No uncertainty quantification tools**

→ prediction of the *worst case scenario* important for security reasons

**Sensitivity analysis (SA)** : study of how changes in the **inputs** of a model affect the **outputs**

**input parameters**

$a$




**output (state)**

$\mathbf{u}$

**Sensitivity :**

$$\frac{\partial \mathbf{u}}{\partial a} = \mathbf{u}_a$$

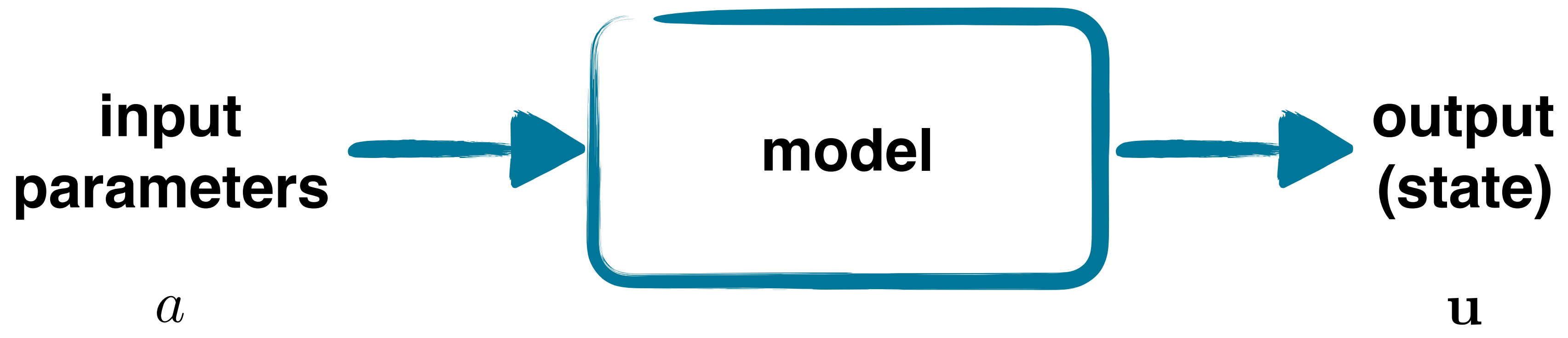
**Continuous sensitivity equation (CSE) method :**

$$\partial_t \mathbf{u} + \mathcal{L}(\mathbf{u}) = \mathbf{f} \quad \Omega,\ t > 0$$

+ initial and boundary conditions.

# Sensitivity Analysis

**Sensitivity analysis (SA)** : study of how changes in the **inputs** of a model affect the **outputs**




**Sensitivity :**

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**Continuous sensitivity equation (CSE) method :**

$$\partial_a \partial_t \mathbf{u} + \partial_a \mathcal{L}(\mathbf{u}) = \partial_a \mathbf{f} \quad \Omega, \ t > 0$$

+ initial and boundary conditions.

# Sensitivity Analysis

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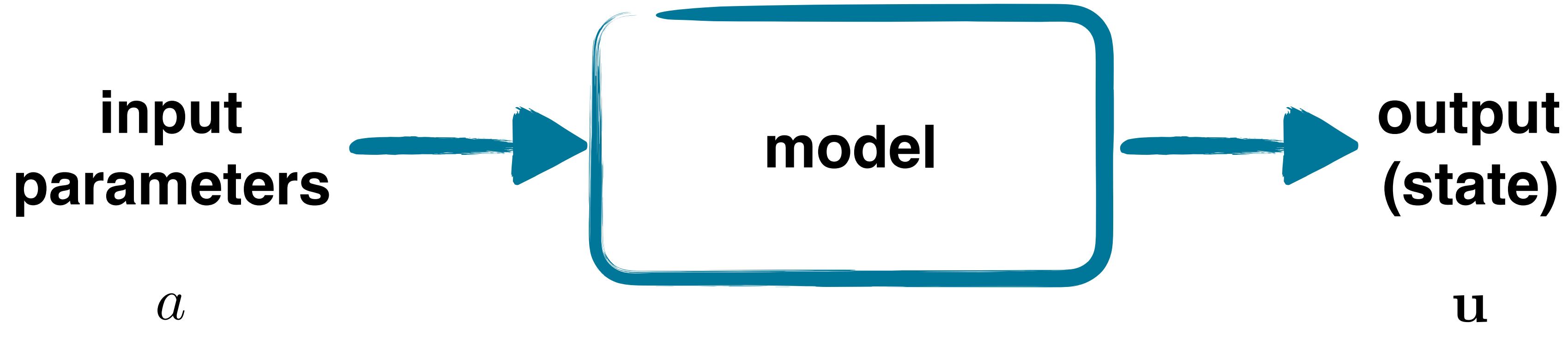


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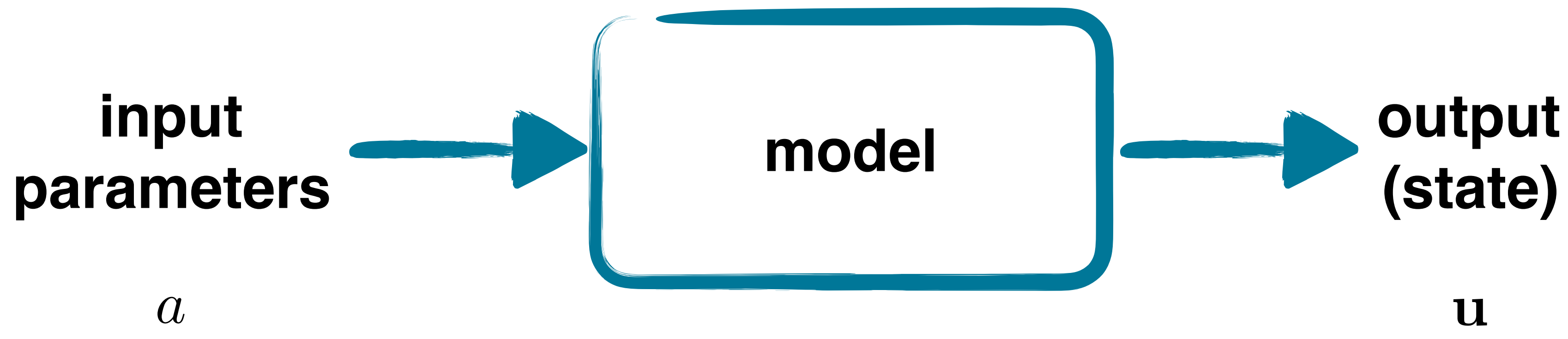
+ initial and boundary conditions.

**Sensitivity analysis (SA)** : study of how changes in the **inputs** of a model affect the **outputs**




**Sensitivity :**

$$\frac{\partial \mathbf{u}}{\partial a} = \mathbf{u}_a$$

**Continuous sensitivity equation (CSE) method :**

$$\partial_t \mathbf{u}_a + \mathcal{L}(\mathbf{u}, \mathbf{u}_a) = \mathbf{f}_a \quad \Omega,\ t > 0$$

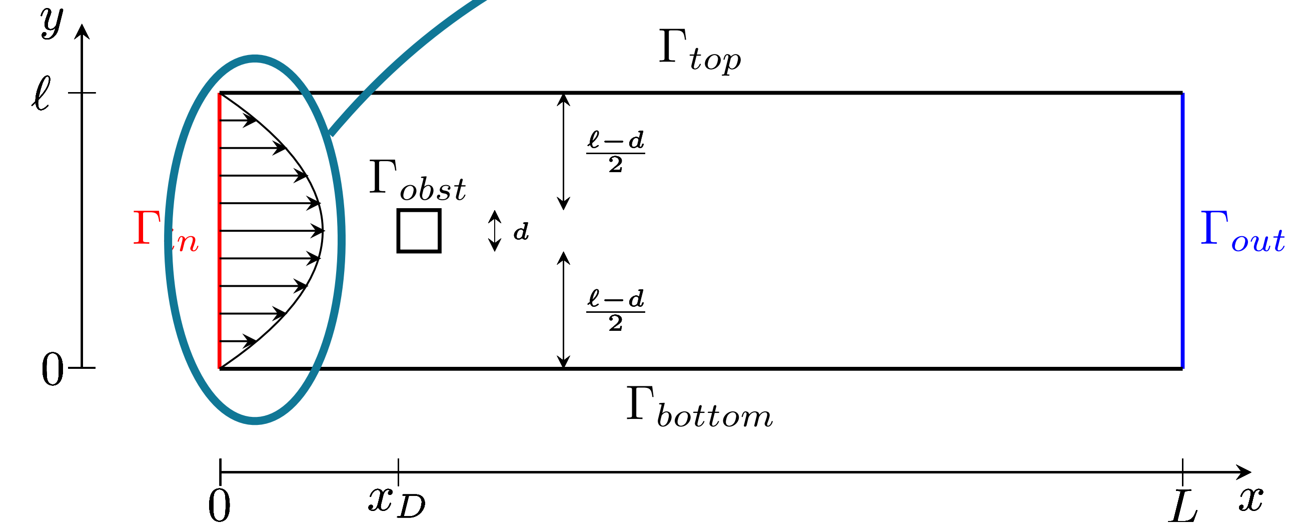
+ initial and boundary conditions.

**The Navier—Stokes equations :**

$$
\begin{cases}
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \Omega, t > 0, \\
\nabla \cdot \mathbf{u} = 0 & \Omega, t > 0, \\
\mathbf{u}(\mathbf{x}, 0) = 0 & \Omega, t = 0, \\
\mathbf{u} = -g(y)\mathbf{n} & \text{on } \Gamma_{in}, \\
\mathbf{u} = 0 & \text{on } \Gamma_w, \\
(\nu \nabla \mathbf{u} - pI)\mathbf{n} = 0 & \text{on } \Gamma_{out}.
\end{cases}
$$

**Domain :**

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$




$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$

**The sensitivity equations :**

$$
\begin{cases}
\partial_t \mathbf{u}_a - \nu \Delta \mathbf{u}_a + (\mathbf{u}_a \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}_a + \nabla p_a = \bar{\mathbf{f}}_a & \Omega, t > 0, \\
\nabla \cdot \mathbf{u}_a = 0 & \Omega, t > 0, \\
\mathbf{u}_a(\mathbf{x}, 0) = 0 & \Omega, t = 0, \\
\mathbf{u}_a = -g_a(y)\mathbf{n} & \text{on } \Gamma_{in}, \\
\mathbf{u}_a = 0 & \text{on } \Gamma_w, \\
(\nu \nabla \mathbf{u}_a - p_a I)\mathbf{n} = 0 & \text{on } \Gamma_{out}.
\end{cases}
$$

$$\bar{\mathbf{f}}_a = \partial_a \mathbf{f} + \partial_a \nu \Delta \mathbf{u}$$

**Remark** : these are known as the Oseen equations.

# State stability

**Proposition 1** *Let $\mathbf{R}_g$ be a sufficiently smooth stationary* [1] *function such that $\nabla \cdot \mathbf{R}_g = 0$ in $\Omega$ and $\mathbf{R}_g = \mathbf{u}$ on $\Gamma_{in} \cup \Gamma_w$. Then, if $\mathbf{u} \cdot \mathbf{n} \geqslant 0$ on $\Gamma_{out}$ the following stability estimate holds for some $\gamma > 0$:*

$$\|\!|\mathbf{u}|\!\|^2 \leqslant \frac{1}{2\nu} \int_0^T \|\tilde{\mathbf{f}}(t)\|^2_{H^{-1}(\Omega)} dt + \gamma \|\mathbf{R}_g\|^2_{H^1(\Omega)},$$

*where $\tilde{\mathbf{f}} = \mathbf{f} + \nu \Delta \mathbf{R}_g - (\mathbf{R}_g \cdot \nabla)\mathbf{R}_g$, and the norm $\|\!|\cdot|\!\|$ is defined as follows:*

$$\|\!|\mathbf{u}|\!\|^2 := \|\mathbf{u}(T)\|^2 + C \int_0^T \|\nabla \mathbf{u}(t)\|^2 dt.$$

$L^2 - \text{norm}$

[1] If the boundary conditions depend on time, $\mathbf{R}_g$ cannot be taken stationary.

**Proposition 2** *Let $\mathbf{R}_{g_a}$ be a sufficiently smooth stationary function such that $\nabla \cdot \mathbf{R}_{g_a} = 0$ in $\Omega$ and $\mathbf{R}_{g_a} = \mathbf{u}_a$ on $\Gamma_{in} \cup \Gamma_w$. Then, if $\mathbf{u} \cdot \mathbf{n} \geqslant 0$ on $\Gamma_{out}$ and if*

$$\exists \kappa_1 = \kappa_1(\mathbf{u}, \Omega) \; : \; -\int_\Omega [(\mathbf{u}_a \cdot \nabla)\mathbf{u}] \cdot \mathbf{u}_a \leqslant \kappa_1 \|\nabla \mathbf{u}_a\|^2$$

*the following stability estimate holds for some $\gamma_1 > 0$:*

$$|||\mathbf{u}_a|||^2 \leqslant \frac{1}{2\nu} \int_0^T \|\tilde{\mathbf{f}}_a(t)\|^2_{H^{-1}(\Omega)} dt + \gamma_1 \|\mathbf{R}_{g_a}\|^2_{H^1(\Omega)},$$

*where $\tilde{\mathbf{f}}_a = \bar{\mathbf{f}}_a + \nu \Delta \mathbf{R}_{g_a} - (\mathbf{R}_{g_a} \cdot \nabla)\mathbf{R}_{g_a}$.*

"Morally" it is $\|\nabla \mathbf{u}\|$

$$-\int_\Omega [(\mathbf{u}_a \cdot \nabla)\mathbf{u}] \cdot \mathbf{u}_a \leqslant \|\mathbf{u}_a\|^2_{L^4} \|\nabla \mathbf{u}\| \leqslant C \|\nabla \mathbf{u}\| \|\nabla \mathbf{u}_a\|^2$$

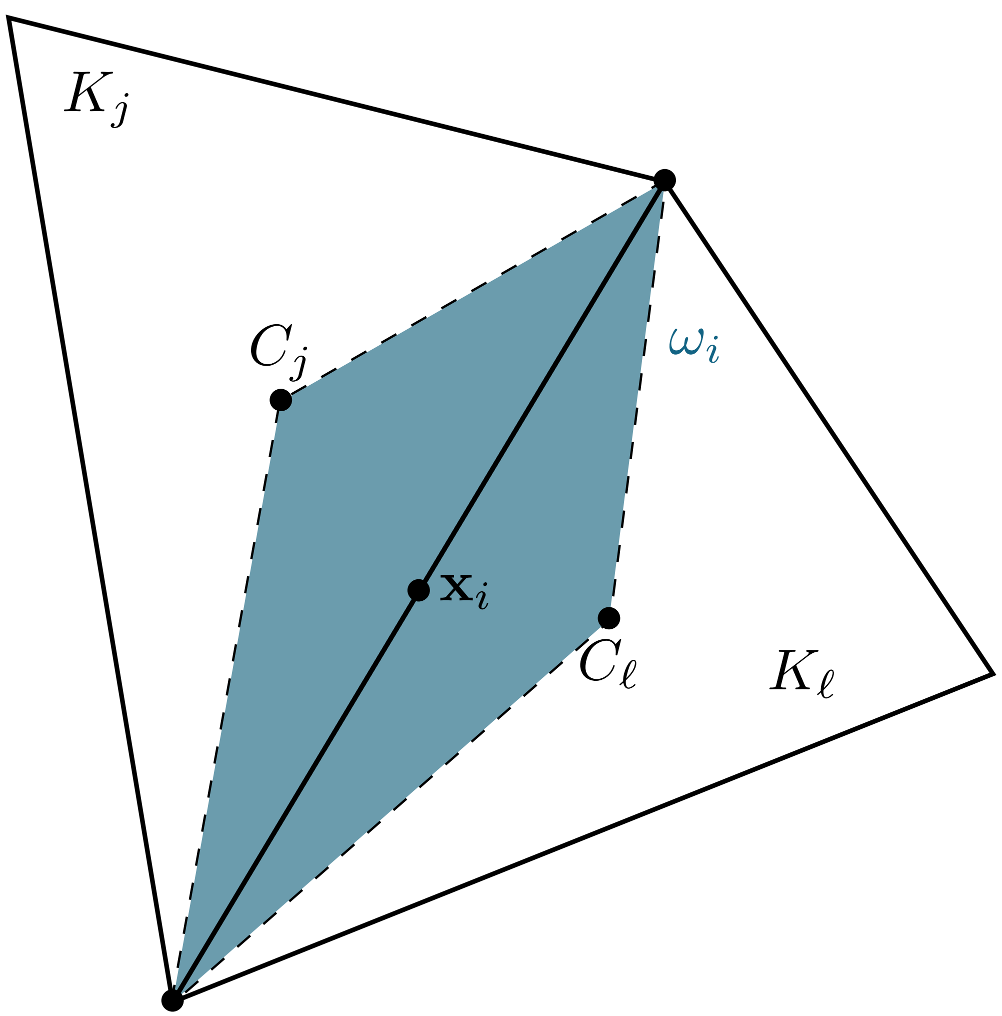
$H^1 \subset L^4$

# Time discretisation

A number of different time schemes are implemented in the code TRUST TrioCFD:

- Forward Euler
- Runge Kutta
  - 2nd, 3rd and 4th order
- Semi-implicit Euler
  - implicit diffusion
  - explicit convection

The code TRUST TrioCFD is based on a **finite elements volumes** method (FEV).




**Ingredients:**

$\mathcal{T}_h$ triangulation of the domain $\Omega$

$K_j \in \mathcal{T}_h$ triangles $j = 1, \ldots, N_T$

$\mathbf{x}_i$ nodes $i = 1, \ldots, N_N$

$\omega_i$ control volume

**Spaces:**

$$Q_h = \{q_h : \forall K \in \mathcal{T}_h,\ q_h|_K \in P_0(K)\},$$
$$V_h = \{w_h \text{ continuous in } \mathbf{x}_i : \forall K \in \mathcal{T}_h,\ w_h|_K \in P_1(K)\},$$
$$\mathbf{V}_h = \{\mathbf{w}_h = (w_x, w_y)^t : w_x,\ w_y \in V_h\}.$$

**Basis functions** : $\varphi_i(\mathbf{x}_j) = \delta_{i,j}$ for $V_h$

$\chi_K$ for $Q_h$

**Remark** : $V_h \not\subset H^1(\Omega)$

We integrate the mass equation and its sensitivity over the triangles and the momentum equation and its sensitivity over the control volumes :

$$
\begin{aligned}
\int_{\partial K_j \setminus \Gamma_D} \mathbf{u}_h \cdot \mathbf{n} &= -\int_{\partial K_j \cap \Gamma_D} \mathbf{u}_h \cdot \mathbf{n} \quad \forall K_j \in \mathcal{T}_h, \\
-\int_{\partial \omega_i \setminus \Gamma_N} (\nu \nabla \mathbf{u}_h - p_h I)\mathbf{n} + \int_{\partial \omega_i} (\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n} &= \int_{\omega_i} \mathbf{f} \quad \forall \omega_i, \\
\int_{\partial K_j \setminus \Gamma_D} \mathbf{u}_{a,h} \cdot \mathbf{n} &= -\int_{\partial K_j \cap \Gamma_D} \mathbf{u}_{a,h} \cdot \mathbf{n} \quad \forall K_j \in \mathcal{T}_h, \\
-\int_{\partial \omega_i \setminus \Gamma_N} (\nu \nabla \mathbf{u}_{a,h} - p_{a,h} I)\mathbf{n} + \int_{\partial \omega_i} (\mathbf{u}_{a,h} \otimes \mathbf{u}_h + \mathbf{u}_h \otimes \mathbf{u}_{a,h})\mathbf{n} &= \int_{\omega_i} \overline{\mathbf{f}}_a \quad \forall \omega_i.
\end{aligned}
$$

Using the basis functions of the spaces previously introduced one has :

$$
\begin{aligned}
\mathbf{u}_h(\mathbf{x}) &= \sum_{i=1}^{N_N} \mathbf{u}_h(\mathbf{x}_i)\varphi_i(\mathbf{x}), & p_h(\mathbf{x}) &= \sum_{j=1}^{N_T} p_h(K_j)\chi_{K_j}, \\
\mathbf{u}_{a,h}(\mathbf{x}) &= \sum_{i=1}^{N_N} \mathbf{u}_{a,h}(\mathbf{x}_i)\varphi_i(\mathbf{x}), & p_{a,h}(\mathbf{x}) &= \sum_{j=1}^{N_T} p_{a,h}(K_j)\chi_{K_j}.
\end{aligned}
$$

# Spatial discretisation

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\setminus\Gamma_D}\varphi_i\mathbf{n} = -\sum_{i\in\mathcal{D}}\mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}}\int_{\partial\omega_i}(\nu\mathbf{u}_h(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n} + \sum_{j=1}^{N_T}p_h(K_j)\int_{\partial\omega_i}\chi_{K_j}\mathbf{n} + \int_{\partial\omega_i}\mathbf{u}_h(\mathbf{u}_h\cdot\mathbf{n}) = \int_{\omega_i}\mathbf{f} \qquad \forall\omega_i,$$

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \mathbf{u}_{a,h}(\mathbf{x}_i)\cdot\int_{\partial K_j\setminus\Gamma_D}\varphi_i\mathbf{n} = -\sum_{i\in\mathcal{D}}\mathbf{u}_{a,h}(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}}\int_{\partial\omega_i}(\nu\mathbf{u}_{a,h}(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n} + \sum_{j=1}^{N_T}p_{a,h}(K_j)\int_{\partial\omega_i}\chi_{K_j}\mathbf{n} + \int_{\partial\omega_i}\mathbf{u}_{a,h}(\mathbf{u}_h\cdot\mathbf{n}) + \mathbf{u}_h(\mathbf{u}_{a,h}\cdot\mathbf{n}) = \int_{\omega_i}\bar{\mathbf{f}}_a \quad \forall\omega_i,$$

# Spatial discretisation

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \overbrace{\mathbf{u}_h(\mathbf{x}_i)}^{\mathbf{U}_i} \cdot \overbrace{\int_{\partial K_j\setminus\Gamma_D} \varphi_i \mathbf{n}}^{B_{j,i}} = \overbrace{-\sum_{i\in\mathcal{D}} \mathbf{u}_h(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n}}^{D_j} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}} \overbrace{\int_{\partial\omega_i}(\nu\,\mathbf{u}_h(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n}}^{A_{i,j}} + \sum_{j=1}^{N_T} \overbrace{p_h(K_j)}^{P_j} \overbrace{\int_{\partial\omega_i}\chi_{K_j}\mathbf{n}}^{C_{i,j}} + \overbrace{\int_{\partial\omega_i}\mathbf{u}_h(\mathbf{u}_h\cdot\mathbf{n})}^{[L(\mathbf{U})\mathbf{U}]_i} = \overbrace{\int_{\omega_i}\mathbf{f}}^{F_i} \qquad \forall\omega_i,$$

$$\sum_{i\in\mathcal{I}\cup\mathcal{N}} \overbrace{\mathbf{u}_{a,h}(\mathbf{x}_i)}^{\mathbf{U}_{a,i}} \cdot \overbrace{\int_{\partial K_j\setminus\Gamma_D} \varphi_i \mathbf{n}}^{B_{j,i}} = \overbrace{-\sum_{i\in\mathcal{D}} \mathbf{u}_{a,h}(\mathbf{x}_i)\cdot\int_{\partial K_j\cap\Gamma_D}\varphi_i\mathbf{n}}^{D_{a,j}} \qquad \forall K_j\in\mathcal{T}_h,$$

$$-\sum_{j\in\mathcal{I}} \overbrace{\int_{\partial\omega_i}(\nu\,\mathbf{u}_{a,h}(\mathbf{x}_j)\otimes\nabla\varphi_j)\,\mathbf{n}}^{A_{i,j}} + \sum_{j=1}^{N_T} \overbrace{p_{a,h}(K_j)}^{P_{a,j}} \overbrace{\int_{\partial\omega_i}\chi_{K_j}\mathbf{n}}^{C_{i,j}} + \overbrace{\int_{\partial\omega_i}\mathbf{u}_{a,h}(\mathbf{u}_h\cdot\mathbf{n})+\mathbf{u}_h(\mathbf{u}_{a,h}\cdot\mathbf{n})}^{[L(\mathbf{U}_a)\mathbf{U}+L(\mathbf{U})\mathbf{U}_a]_i} = \overbrace{\int_{\omega_i}\bar{\mathbf{f}}_a}^{F_{a,i}} \quad \forall\omega_i,$$

# Spatial discretisation

$$A\mathbf{U} + B^t P + L(\mathbf{U})\mathbf{U} = \mathbf{F}$$
$$B\mathbf{U} = D$$
$$A\mathbf{U}_a + B^t P_a + L(\mathbf{U}_a)\mathbf{U} + L(\mathbf{U})\mathbf{U}_a = \mathbf{F}_a$$
$$B\mathbf{U}_a = D_a$$

We consider explicit Euler in time with homogeneous Dirichlet b.c. for simplicity's sake:

$$\begin{cases} M\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t} = -A\mathbf{U}^n - L(\mathbf{U}^n)\mathbf{U}^n - B^t P^{n+1} + F^n \\ B\mathbf{U}^{n+1} = 0 \end{cases}$$

$$M\frac{\mathbf{U}^* - \mathbf{U}^n}{\Delta t} = -A\mathbf{U}^n - L(\mathbf{U}^n)\mathbf{U}^n - B^t P^n + F^n$$

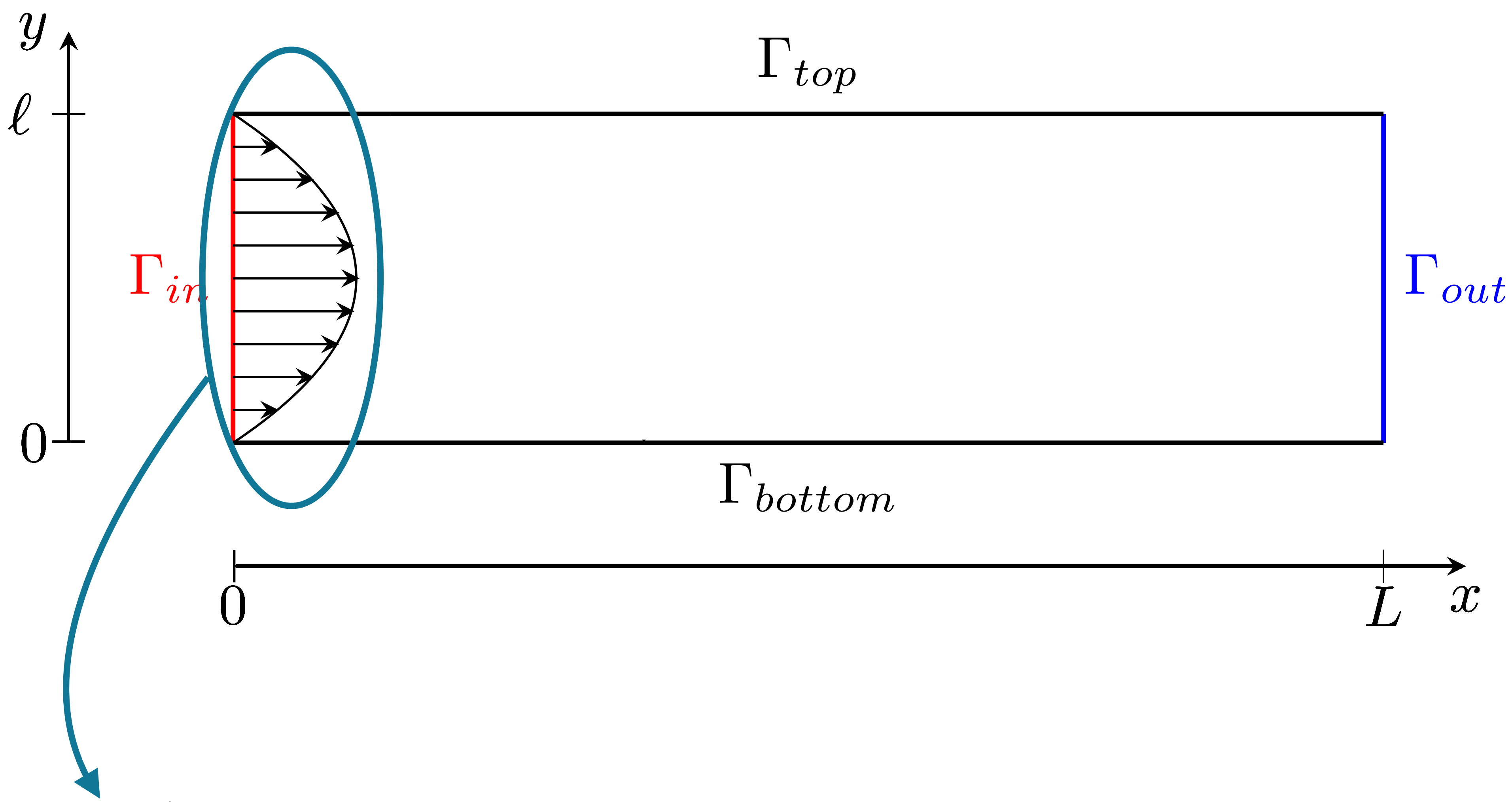
$$(BM^{-1}B^t)\delta P = \frac{1}{\Delta t} B\mathbf{U}^*$$

$$\mathbf{U}^{n+1} = \mathbf{U}^* - \Delta t M^{-1} B^t \delta P$$

$$P^{n+1} = P^n + \delta P$$

## Test case description

**Domain :**

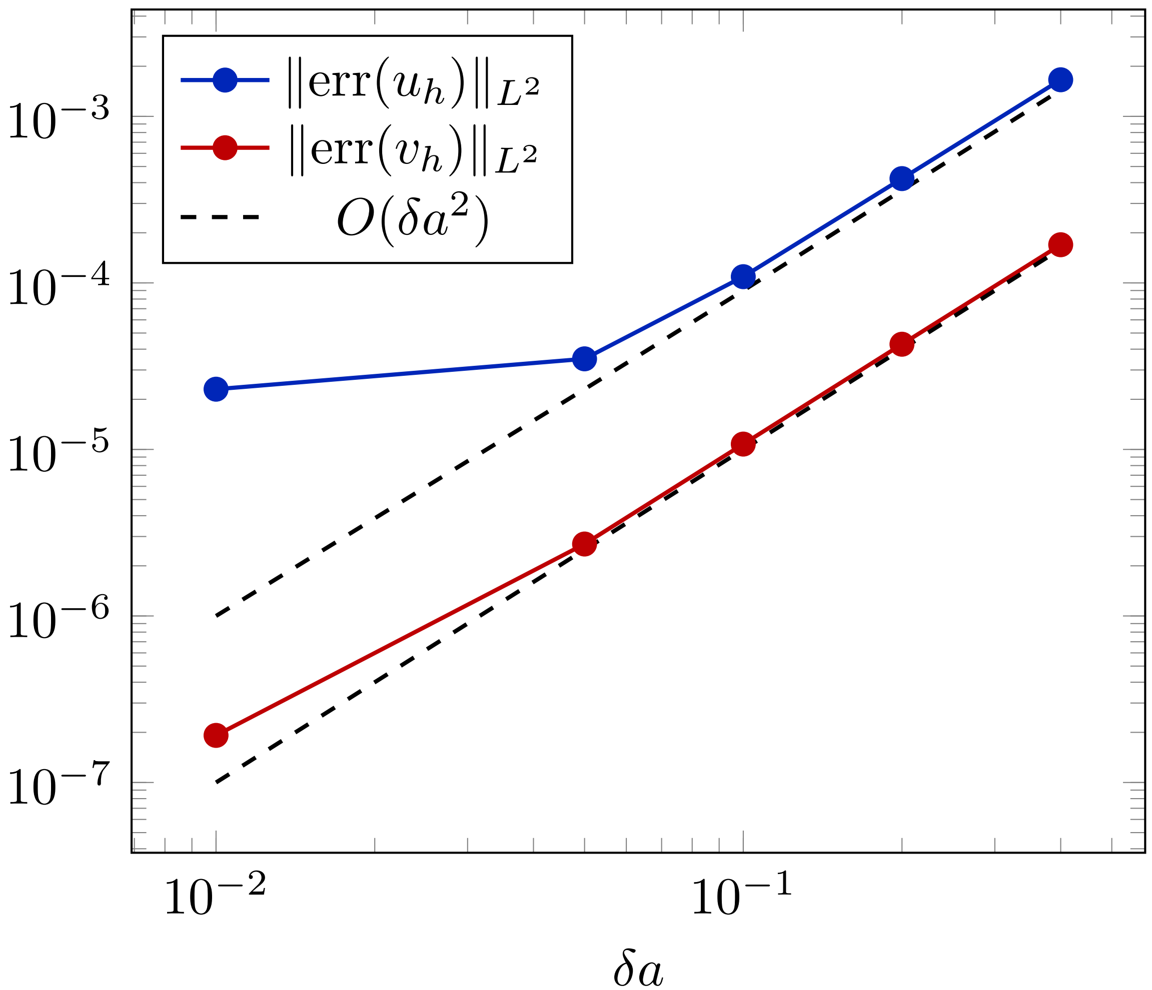


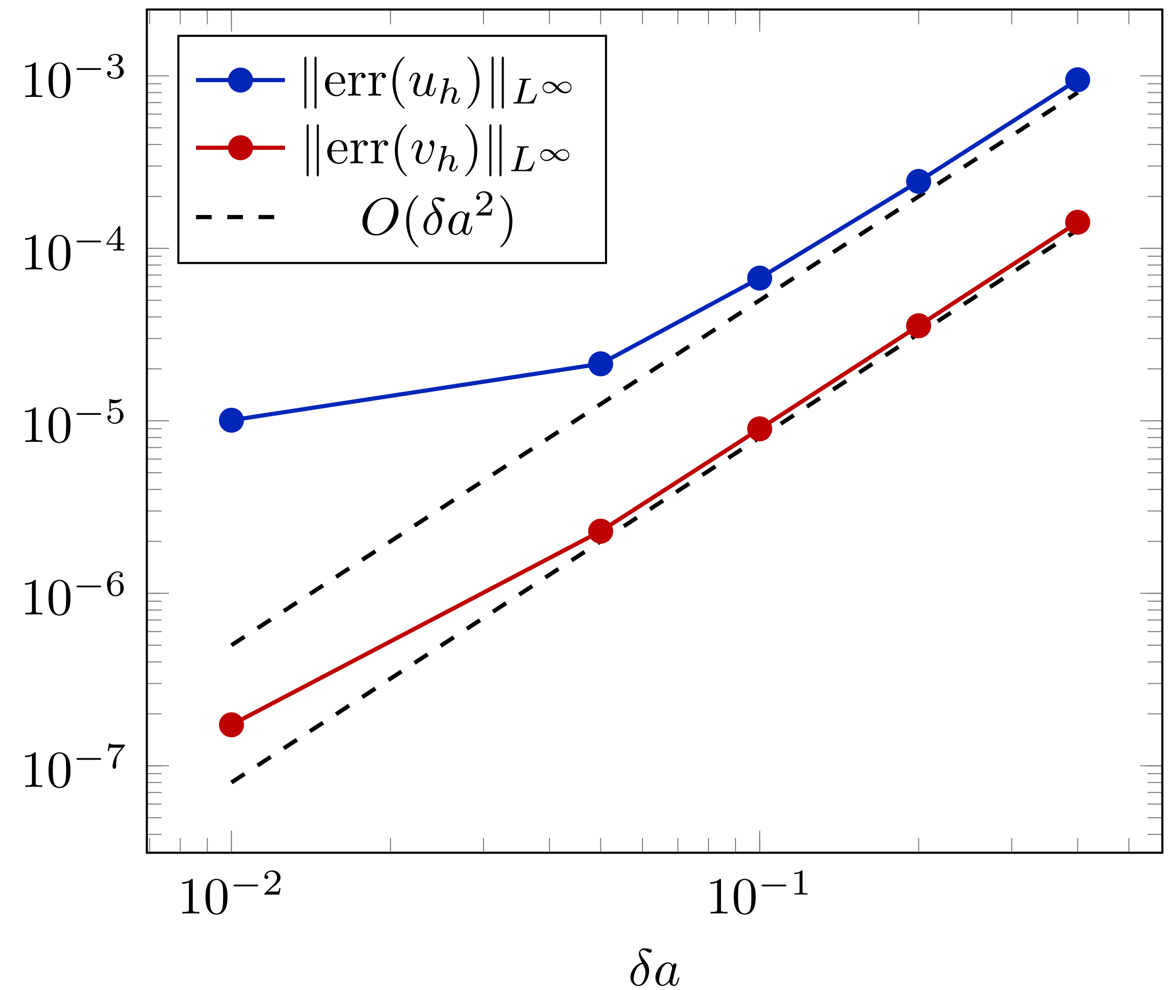

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$

Uncertain parameter $a = A$

### Results

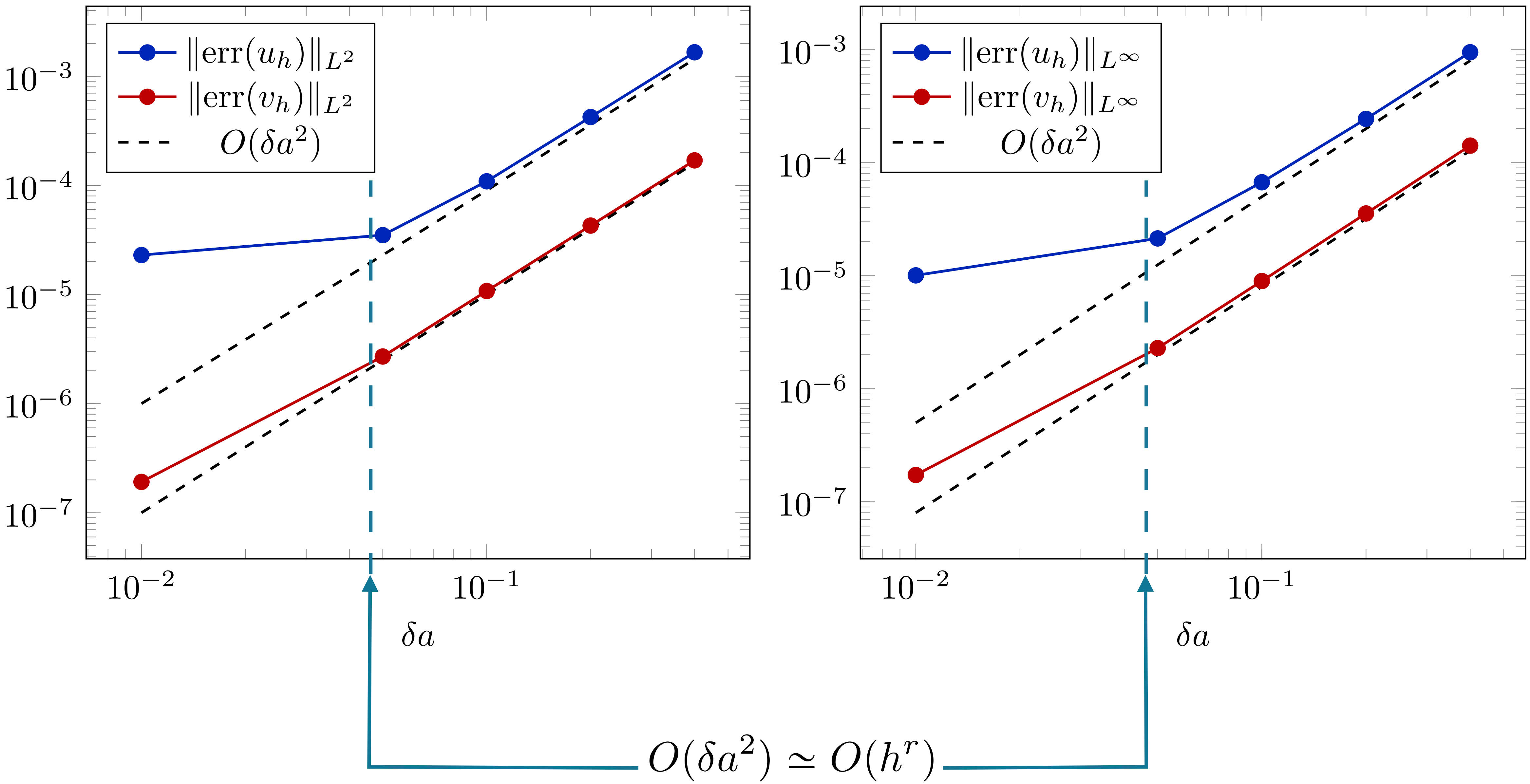
$$\mathrm{err}(\mathbf{u}) = \mathbf{u}(x,T;a+\delta a) - \mathbf{u}(x,T;a) - \delta a \mathbf{u}_a(x,T;a) \simeq O(\delta a^2)$$



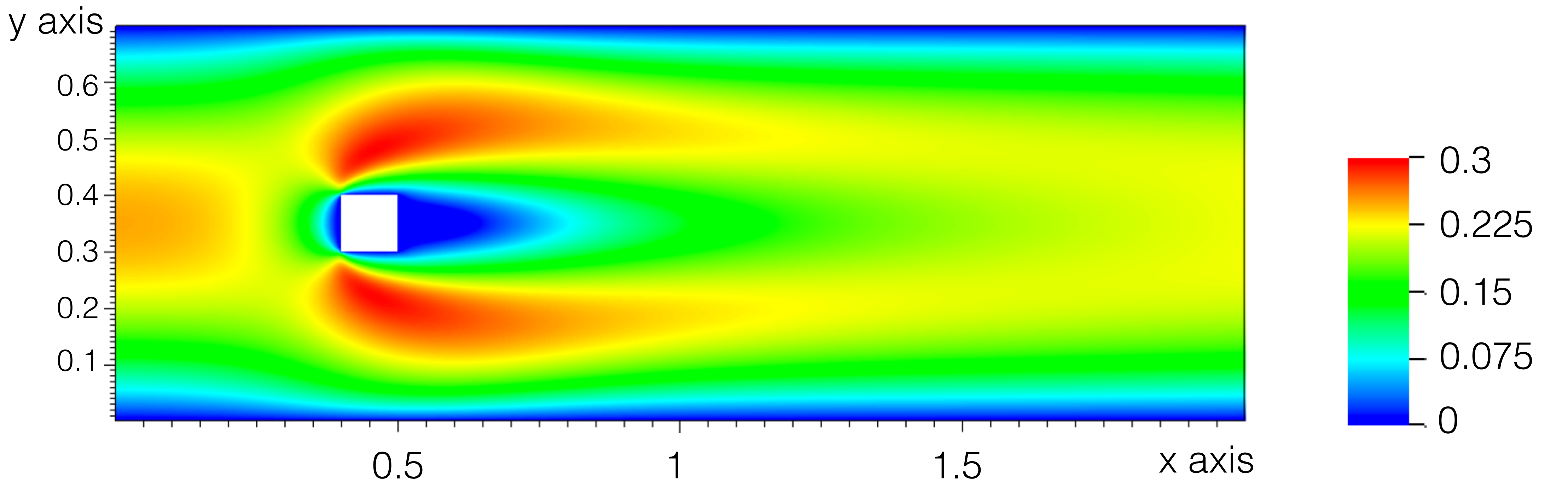
### Results

$$\mathrm{err}(\mathbf{u}) = \mathbf{u}(x,T;a+\delta a) - \mathbf{u}(x,T;a) - \delta a\,\mathbf{u}_a(x,T;a) \simeq O(\delta a^2)$$

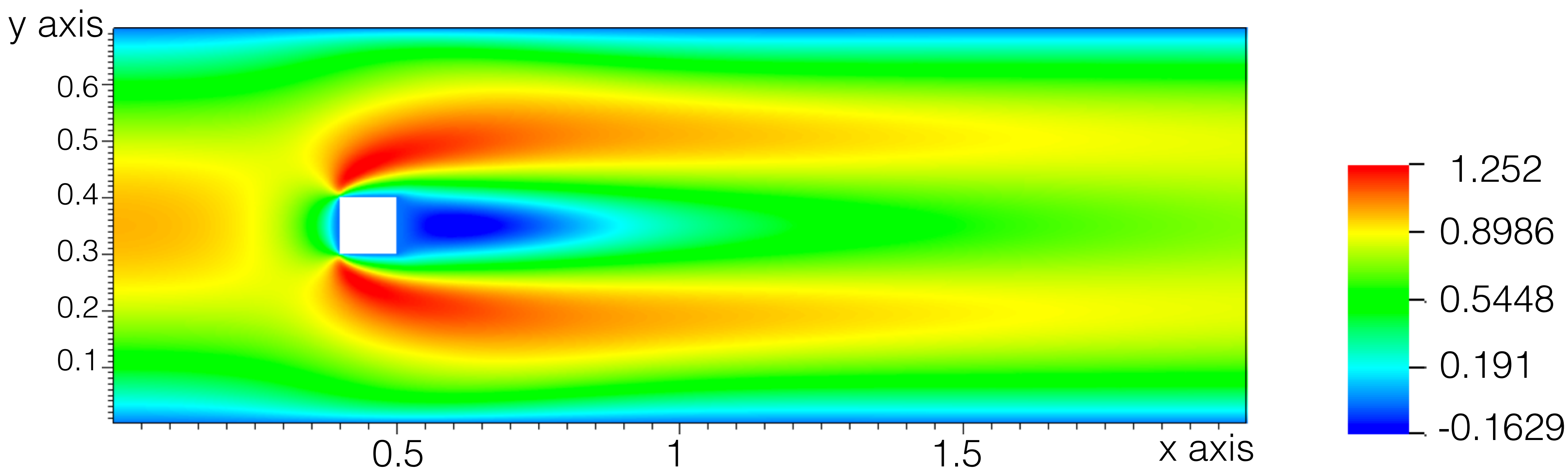


Steady case : x-component of the velocity and its sensitivity

**State**

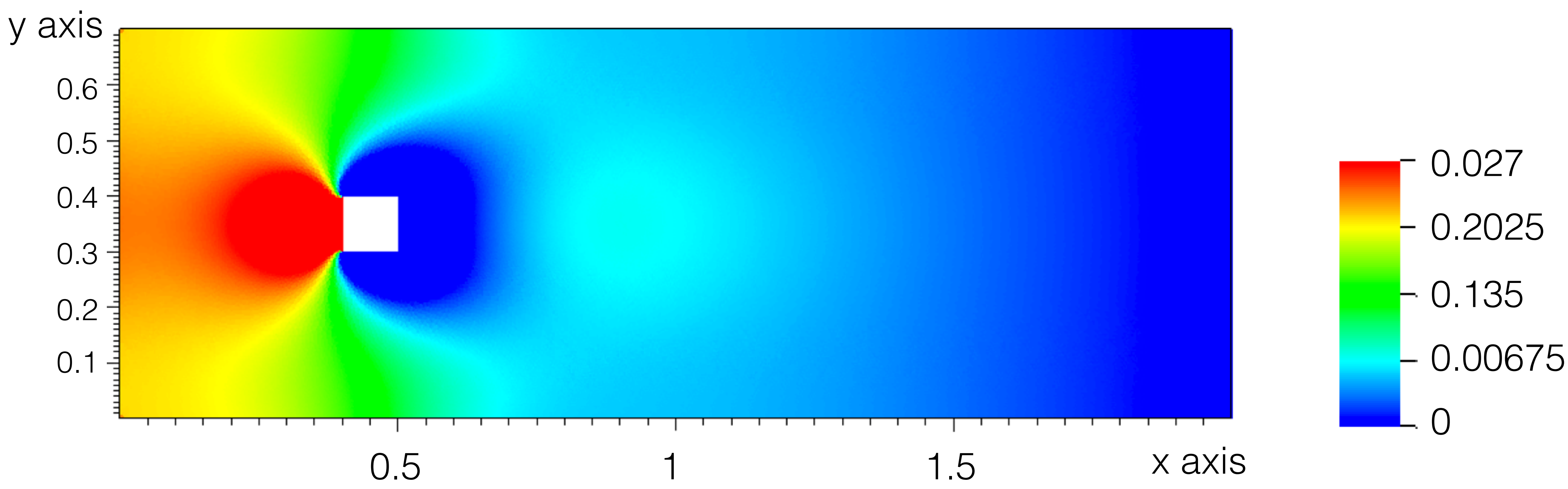



**Sensitivity**

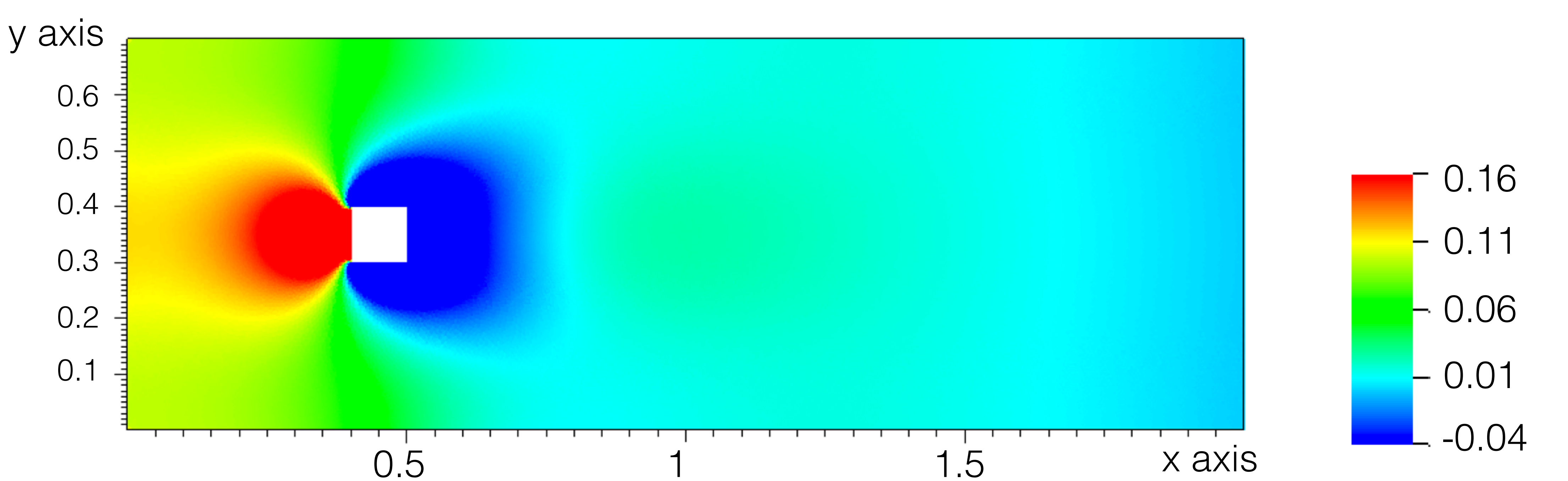


Steady case : pressure and its sensitivity

**State**




**Sensitivity**

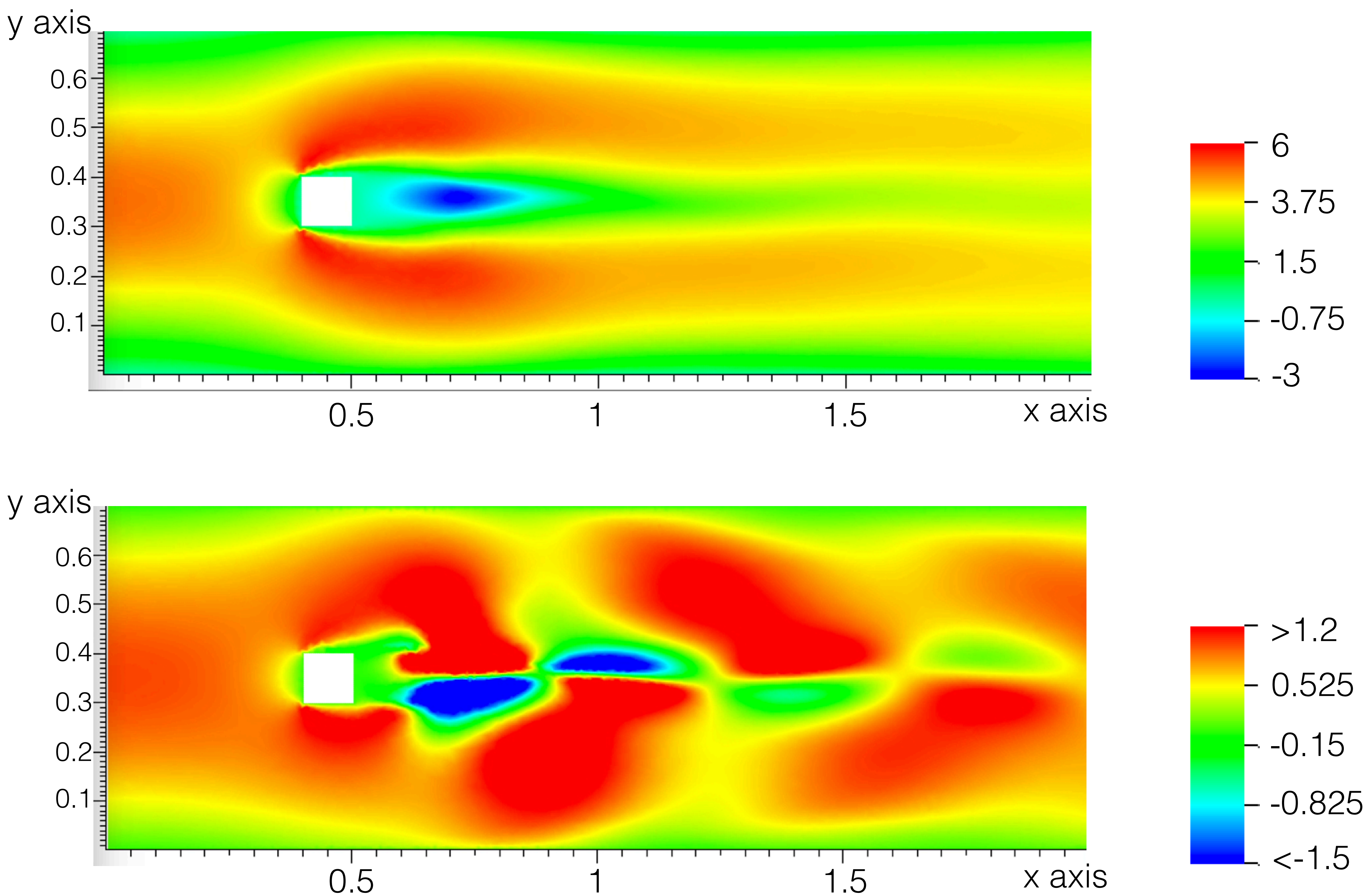


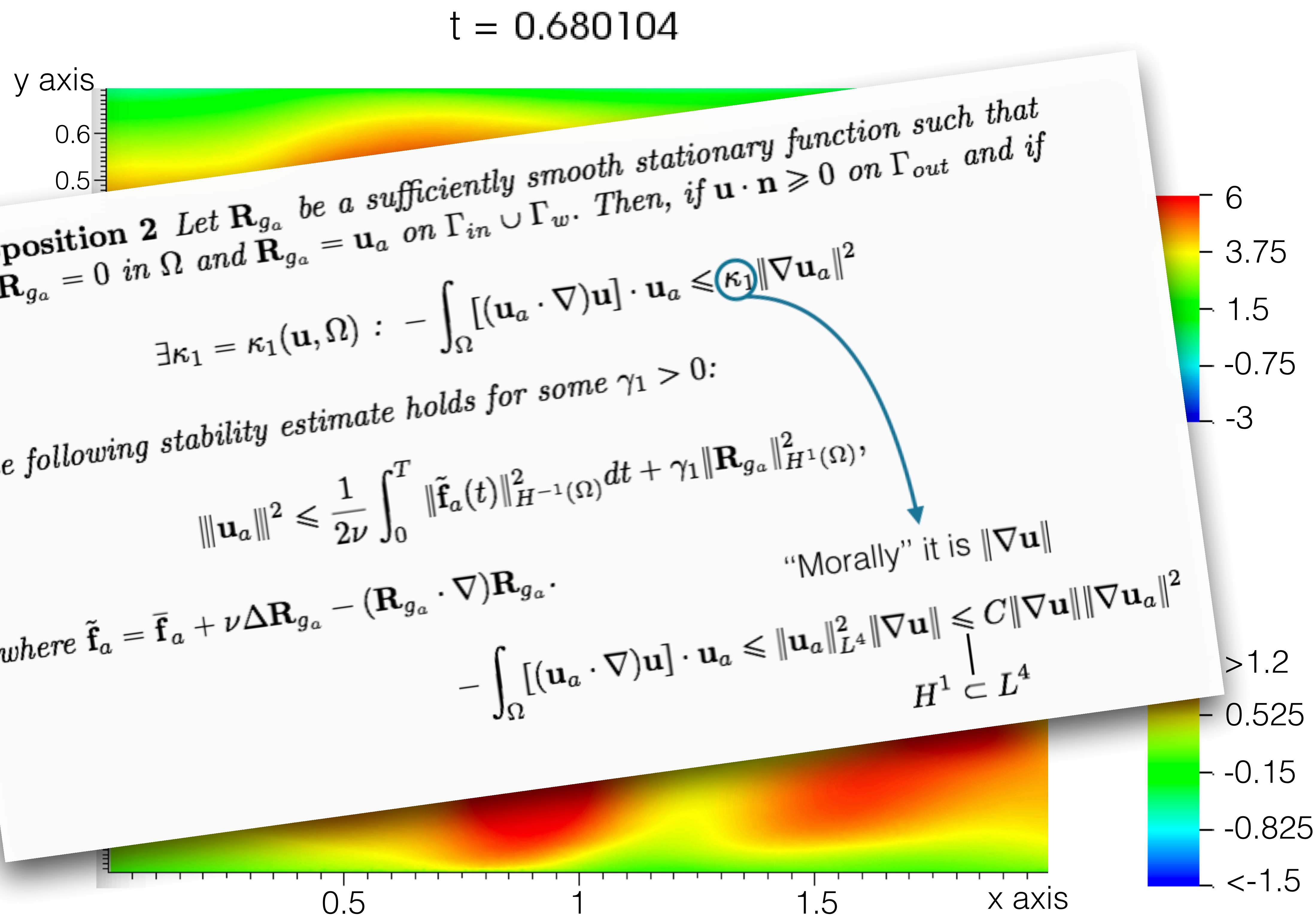
# Numerical results

t = 0.680104

State

Sensitivity






**Proposition 2** *Let $\mathbf{R}_{g_a}$ be a sufficiently smooth stationary function such that $\nabla\cdot\mathbf{R}_{g_a}=0$ in $\Omega$ and $\mathbf{R}_{g_a}=\mathbf{u}_a$ on $\Gamma_{in}\cup\Gamma_w$. Then, if $\mathbf{u}\cdot\mathbf{n}\geqslant 0$ on $\Gamma_{out}$ and if*

$$\exists\kappa_1=\kappa_1(\mathbf{u},\Omega)\ :\ -\int_\Omega[(\mathbf{u}_a\cdot\nabla)\mathbf{u}]\cdot\mathbf{u}_a\leqslant\kappa_1\|\nabla\mathbf{u}_a\|^2$$

*the following stability estimate holds for some $\gamma_1>0$:*

$$|||\mathbf{u}_a|||^2\leqslant\frac{1}{2\nu}\int_0^T\|\tilde{\mathbf{f}}_a(t)\|^2_{H^{-1}(\Omega)}dt+\gamma_1\|\mathbf{R}_{g_a}\|^2_{H^1(\Omega)},$$

*where* $\tilde{\mathbf{f}}_a=\bar{\mathbf{f}}_a+\nu\Delta\mathbf{R}_{g_a}-(\mathbf{R}_{g_a}\cdot\nabla)\mathbf{R}_{g_a}$.

"Morally" it is $\|\nabla\mathbf{u}\|$

$$-\int_\Omega[(\mathbf{u}_a\cdot\nabla)\mathbf{u}]\cdot\mathbf{u}_a\leqslant\|\mathbf{u}_a\|^2_{L^4}\|\nabla\mathbf{u}\|\leqslant C\|\nabla\mathbf{u}\|\|\nabla\mathbf{u}_a\|^2$$

$H^1\subset L^4$

Let **a** be a gaussian random vector, with the following average and variance:

$$\mu_{\mathbf{a}} = \begin{bmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_M} \end{bmatrix}, \qquad \sigma_{\mathbf{a}} = \begin{bmatrix} \sigma^2_{a_1} & \mathrm{cov}(a_1, a_2) & \dots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_1, a_2) & \sigma^2_{a_2} & \dots & \mathrm{cov}(a_2, a_M) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_1, a_M) & \dots & & \sigma^2_{a_M} \end{bmatrix},$$

Aim: determine a **confidence interval**

$$CI_X = [\mu_X - d(\sigma_X), \mu_X + d(\sigma_X)]$$
$$P(X \in CI_X) \geqslant 1 - \alpha$$

**Monte Carlo** approach: *N* samples of the state $X_k$

$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma^2_X = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

# Uncertainty quantification

**Sensitivity** approach: let us consider the following first order Taylor expansion

$$X(\mathbf{a}) = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} (a_i - \mu_{a_i}) X_{a_i}(\mu_{\mathbf{a}}) + o(\|\mathbf{a} - \mu_{\mathbf{a}}\|^2)$$

Then, computing, the average one has:

$$\mu_X = E[X(\mathbf{a})] = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}}) E[a_i - \mu_{a_i}] = X(\mu_{\mathbf{a}}),$$

And for the variance:

$$\sigma_X^2 = E[(X(\mathbf{a}) - \mu_X)^2] = E\left[\left(\sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})(a_i - \mu_{a_i})\right)^2\right] =$$

$$= \sum_{i=1}^{M} X_{a_i}^2(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i}(\mu_{\mathbf{a}}) X_{a_j}(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})(a_j - \mu_{a_j})].$$

Therefore, we have the following first order estimates:

$$\mu_X = X(\mu_{\mathbf{a}}), \quad \sigma_X^2 = \sum_{i=1}^{M} X_{a_i}^2 \sigma_{a_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i} X_{a_j} \mathrm{cov}(a_i, a_j).$$

**Monte Carlo approach:**

Provides information on the distribution of the output

Smaller confidence intervals

Computational cost

**SA approach :**

Reasonable computational cost (depending on the number of parameters)

No insight on the distribution of the output

Valid for small variation of the input

*How to compute a confidence interval for an unknown distribution?*

**Chebyshev's inequality:** $P(|X-\mu_X| \geqslant \lambda) \leqslant \dfrac{\sigma_X^2}{\lambda^2} \quad \forall\lambda > 0$

Which implies: $P(X \in (\mu_X - \lambda, \mu_X + \lambda)) \geqslant 1 - \dfrac{\sigma_X^2}{\lambda^2}.$

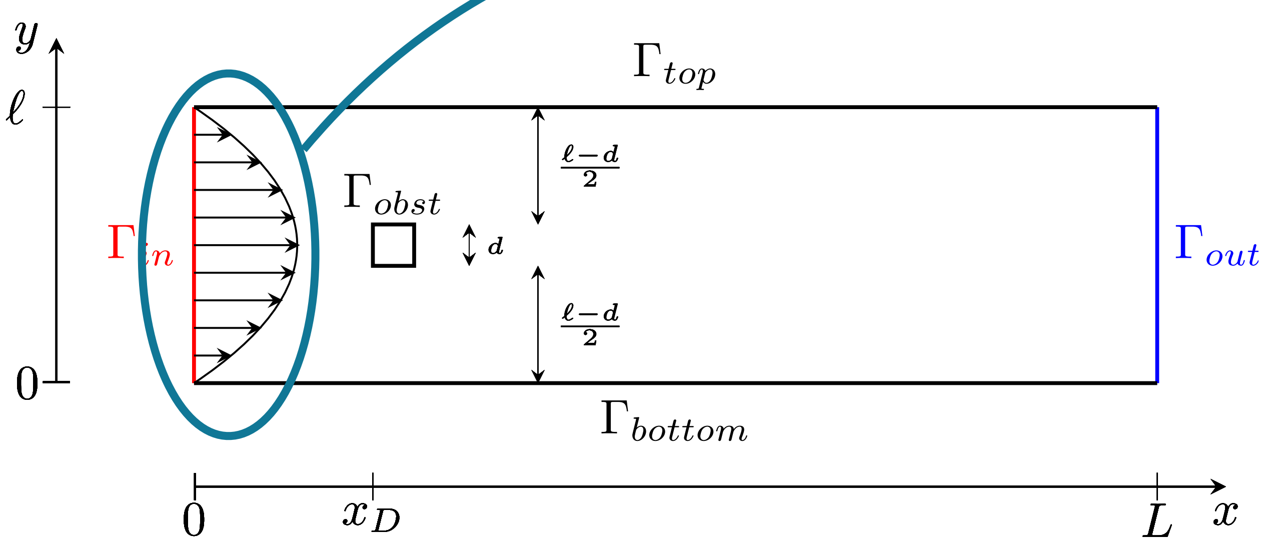
Now, by asking for $1 - \dfrac{\sigma_X^2}{\lambda^2} = 1 - \alpha$ one obtains $\lambda = \dfrac{\sigma_X}{\sqrt{\alpha}}.$

**Remark:** if the variable X was gaussian, to obtain a 95% confidence interval the half amplitude should be $1.96\sigma_X$, whilst with this method we obtain $4.47\sigma_X$.

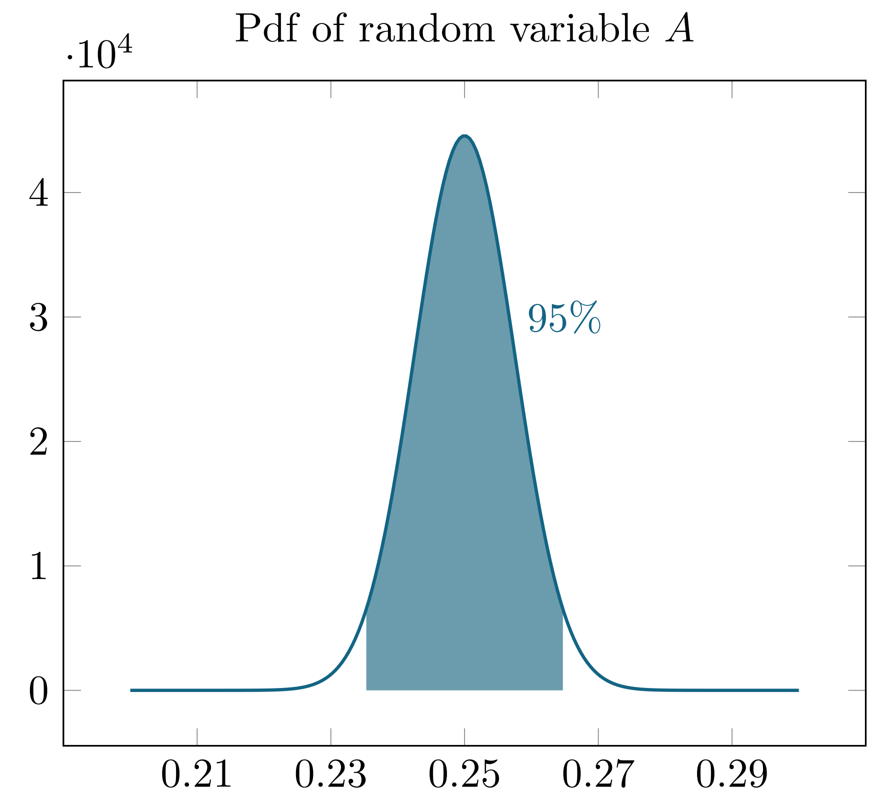
**Domain :**

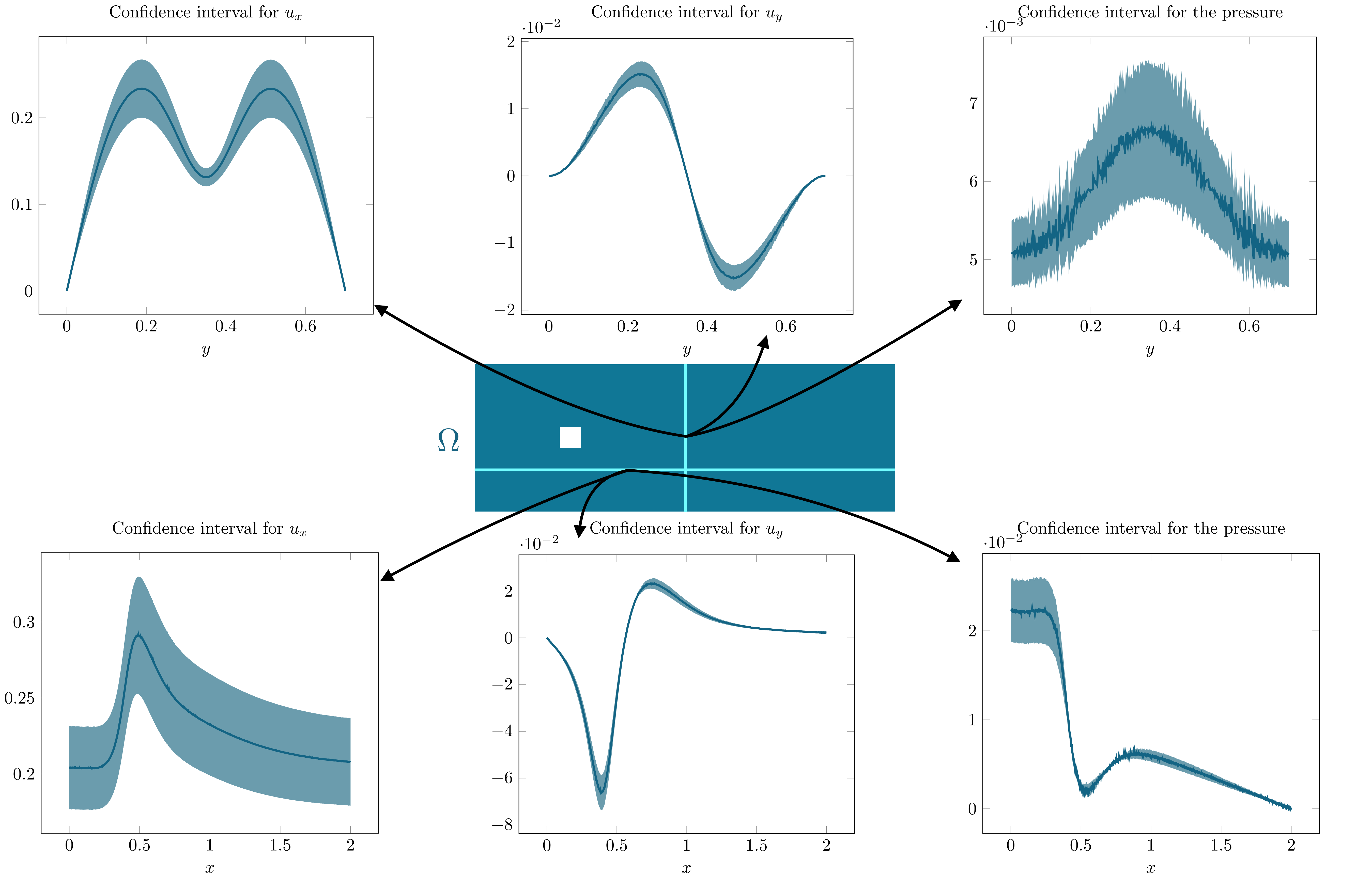
uncertain parameter

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$




$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$




Confidence interval for $u_x$
Confidence interval for $u_y$
Confidence interval for the pressure
$\Omega$
Confidence interval for $u_x$
Confidence interval for $u_y$
Confidence interval for the pressure
$y$
$x$

# Conclusion and perspectives

- Publish the sensitivity module of the code
- Higher order schemes in space
- Extension to 3D
- Coupling with a temperature equation
- More realistic simulations
- Turbulence models

**Thank you**

**for your attention!**