A two-runners model: optimization of running strategies according to the physiological parameters

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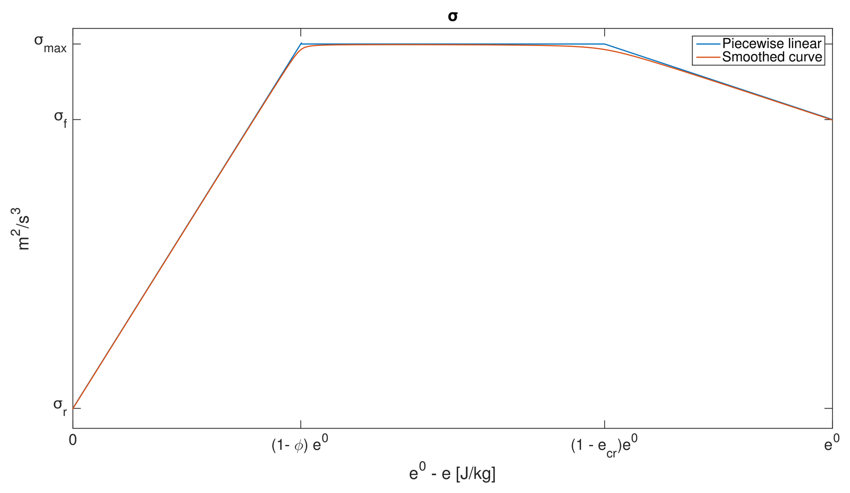
Mathematical models

Single runner model

Aftalion and Bonnans' model [2]:

$$\begin{cases} \dot{x}(t) = v & x(0)=0,\ x(T)=D \\ \dot{v}(t) = f(t) - \frac{v(t)}{\tau} & v(0)=0, \\ \dot{e}(t) = \sigma(e) - f(t)v(t) & e(0)=e^0, \end{cases} \tag{1}$$

- $x(t)$ position at time t; $v(t)$ velocity at time t; $e(t)$ anaerobic energy at time t;
- τ constant coefficient which models the **friction effects**, linear in v
- $\sigma = \sigma(e)$ **oxygen uptake** $\dot{V}O2$ (Figure below).



Physiological constraints:

$$e(t) \geq 0 \quad \forall t \geq 0; \tag{2}$$

$$f \in \mathcal{F} := \{f\ :\ 0 \leq f(t) \leq f_M \quad \forall t \geq 0\}. \tag{3}$$

AIM: solving (1)-(2)-(3) in such a way that, given a distance D, the time T to reach it is minimal.
Optimal control problem:

- f control variable;
- T cost functional to be minimized;

Resulting problem:

$$\min_{f\in\mathcal{F}} T(f) \quad \text{s.t (1)-(2).} \tag{4}$$

Two-runners models

Our model: for $i = 1, 2$

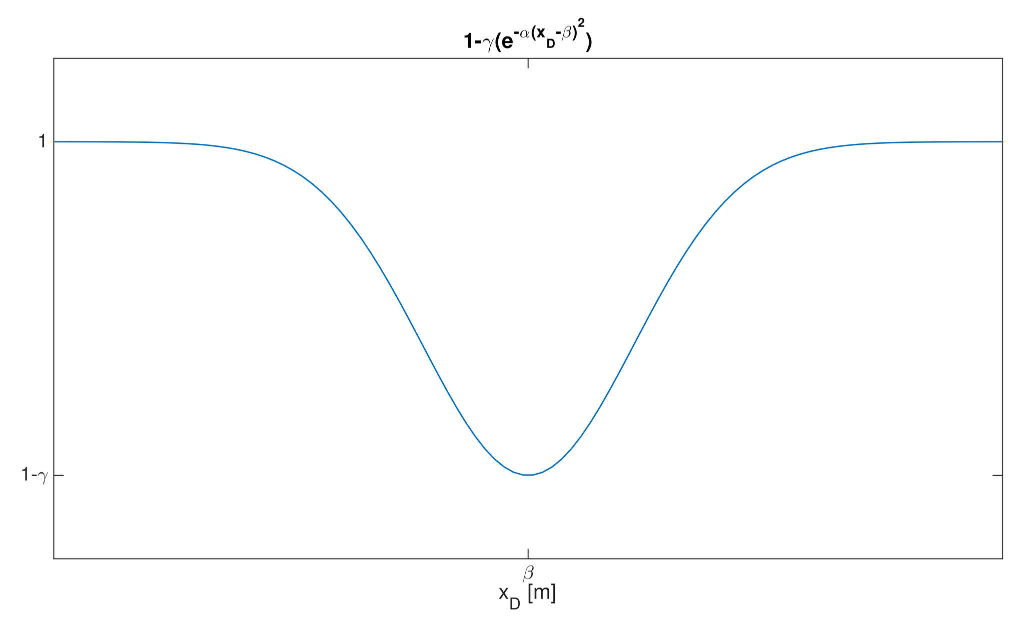
$$\begin{cases} \dot{x}_1 = v_1 & x_1(0)=0 \\ \dot{x}_D = v_2 - v_1 & x_D(0)=0 \\ \dot{v}_1 = f_1 - \frac{v_1}{\tau_1} - cv_1^2(1-\gamma(e^{-\alpha(x_D-\beta)^2})) & v_1(0)=0 \\ \dot{v}_2 = f_2 - \frac{v_2}{\tau_2} - cv_2^2(1-\gamma(e^{-\alpha(x_D+\beta)^2})) & v_2(0)=0 \\ \dot{e}_i = \sigma_i(e_i) - f_i v_i & e_i(0)=e_i^0, \end{cases} \tag{5}$$

- the subscript i refers to the runner;
- $x_D(t) := x_2(t) - x_1(t)$ distance between the runners at time t.

Boundary condition:

$$(x_1(T)-D)(x_2(T)-D) = 0. \tag{6}$$

The physiological constraints (2) and (3) do not change, however the value f_M depends on the runner.



The term $1-\gamma e^{-\alpha(x_D\pm\beta)^2}$, shown in the figure above, encompasses both friction and a **psychological factor**, which consists in trying to follow one's competitor, in order to be able to overtake. It is a potential which has a minimum at distance β behind and decreases global friction because it increases the will to follow. On the other hand, when the other runner is too far, there is no benefit.

Optimization problem

We minimize the following quantity, given a proper constant weight $c_w > 0$:

$$J(f_1, f_2) = T + c_w|x_D(T)|. \tag{7}$$

The resulting problem is:

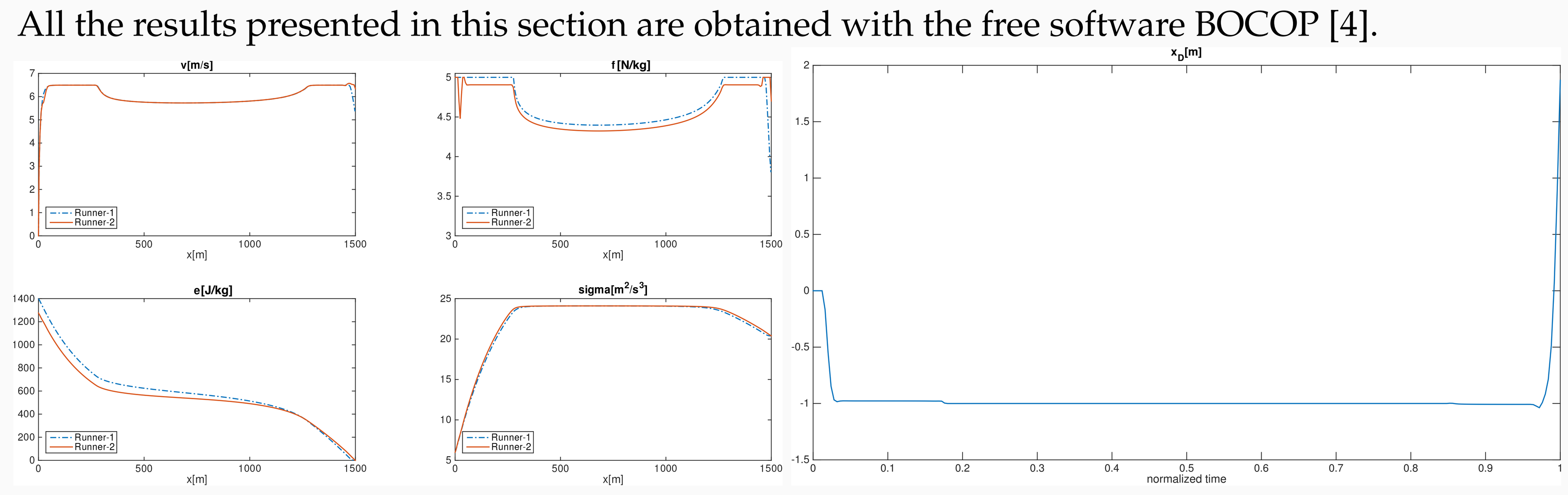
$$\min_{f_i \in \mathfrak{F}_i} J \quad \text{s.t. (5)-(6)-(2),} \tag{8}$$

where $\mathfrak{F}_i$ is the set of the admissible controls which depends on the athlete and is defined as follows:

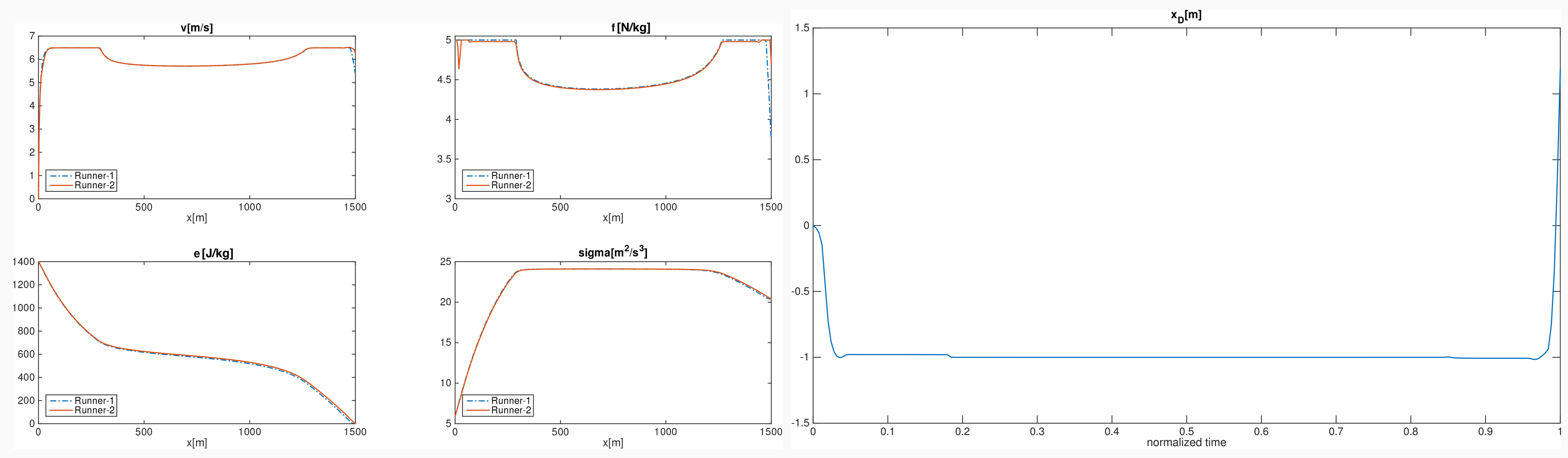
$$\mathfrak{F}_i := \{f : 0 \leq f(t) \leq f_{M,i},\ |\dot{f}(t)| \leq K_i\ \forall t \in (0,T)\}.$$

Numerical results

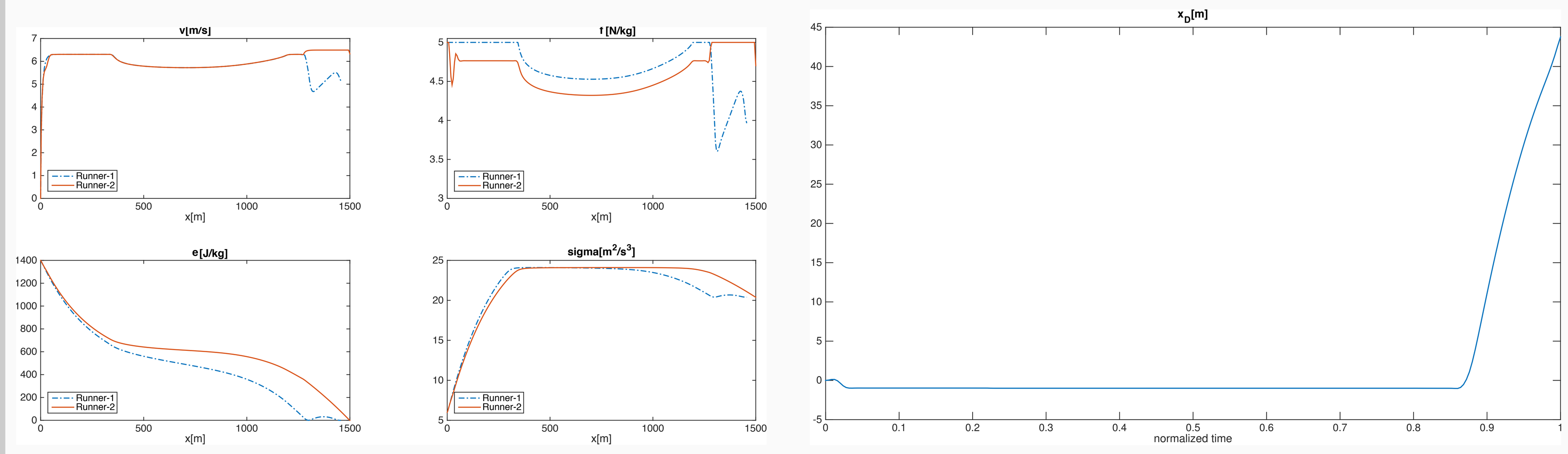
All the results presented in this section are obtained with the free software BOCOP [4].



- Different initial energies: $e_1^0 = 1400J/kg$ and $e_2^0 = 1275J/kg$.
- $x_1(T) = 1498.13m$
- $T = 249.43s$, ($-2s$ w.r.t best performance running alone)
- Overtaking at 99% of the race.



- Different initial τ: $\tau_1 = 1.33s$ $\tau_2 = 1.31s$
- $x_1(T) = 1498.82m$
- $T = 249.536s$, ($-2s$ w.r.t best performance running alone)



- Stronger runner starts behind: $\tau_1 = 1.31s$ $\tau_2 = 1.33s$
- $T = 248.726s$, ($-1s$ w.r.t best performance running alone).
- Overtaking at 87% of the race.

Real races:

- Beijing 2008: overtaking at 84.6% of the race;
- Rome 2014: overtaking at 96.9%;
- Singapore 2015: overtaking at 91.8%.

Conclusion

- new model for a **two-runners problem**, which takes into account **psychological factors**;
- the numerical results show how a runner can **improve his personal best performance** by exploiting the advantage of running behind someone else;
- the major application for Olympic training could be for an athlete to estimate whether he should **stay behind or lead**, and when is **the best time to overtake**;
- the **curvature of the track** and the **parameter identification** are the aim of upcoming papers.

References

[1] A. Aftalion and C. Fiorini, A two-runners model: optimization of running strategies according to the physiological parameters, *submitted*, 2015.

[2] A. Aftalion and J.-F. Bonnans, Optimization of running strategies based on anaerobic energy and variations of velocity, *SIAM Journal on Applied Mathematics*, 74(5):1615-1636, 2014.

[3] A. B. Pitcher, Optimal strategies for a two-runner model of middle-distance running, *SIAM Journal on Applied Mathematics*, 70(4):1032-1046, 2009.

[4] F. Bonnans, D. Giorgi, V. Grelard, S. Maindrault, and P. Martinon, BOCOP - A toolbox for optimal control problems.

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