Analyse de sensibilité pour des modèles de mécanique des fluides

Séminaire d'analyse numérique
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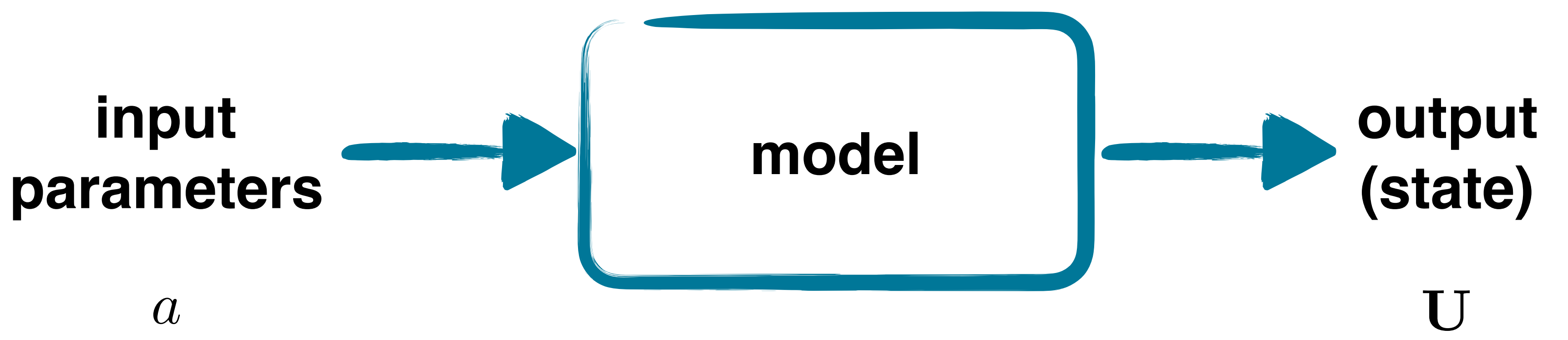
Outline of the talk

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Riemann problem for the Euler equations and their sensitivity
- Classical numerical schemes
- Anti-diffusive numerical schemes
- Applications

Sensitivity analysis

Sensitivity analysis: study of how changes in the **inputs** of a model affect the **outputs**



Sensitivity: $$\frac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$$

- Optimization [5]

Problem: $\min_{a\in\mathcal{A}} J(\mathbf{U})$, where $J(\mathbf{U}) = \frac{1}{2}b(\mathbf{U},\mathbf{U})$ and b is bilinear.

Classical optimization techniques call for the differentiation of the cost function:

$$a^{new} = a^{old} - \alpha\frac{\partial J(\mathbf{U})}{\partial a} \qquad \frac{\partial J(\mathbf{U})}{\partial a} = b(\mathbf{U},\mathbf{U}_a)$$

[5] Borggaard, J., Burns, J. (1997). A PDE sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, *136*(2), 366-384.
[6] Duvigneau, R., Pelletier, D. (2006). A sensitivity equation method for fast evaluation of nearby flows and uncertainty analysis for shape parameters. *International Journal of Computational Fluid Dynamics*, *20*(7), 497-512.
[7] Delenne, C. (2014). Propagation de la sensibilité dans les modèles hydrodynamiques (HDR, Montpellier II).

Applications

- Optimization [5]
- Quick evaluation of close solutions [6]

$$\mathbf{U}(a + \delta a) = \mathbf{U}(a) + \delta a \mathbf{U}_a(a) + o(\delta a^2)$$

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- Optimization [5]
- Quick evaluation of close solutions [6]
- Uncertainty quantification [7]

First order estimates $\mu \quad \mathbf{U}(\mu_a)$

$$\sigma^2 \quad \mathbf{U}_a(\mu_a)^T \mathbf{U}_a(\mu_a)\sigma_a^2$$

[5] Borggaard, J., Burns, J. (1997). A PDE sensitivity equation method for optimal aerodynamic design. *Journal of Computational Physics*, *136*(2), 366-384.
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Two approaches

Discretise then differentiate	Differentiate then discretise
analytical state model	analytical state model
↓ discretise	↓ differentiate
discrete state model	analytical sensitivity model
↓ differentiate	↓ discretise
discrete sensitivity model	discrete sensitivity model

analytical sensitivity model

no discretisation of computational facilitators

could lead to inconsistent gradients

Standard techniques of sensitivity analysis

Standard techniques of sensitivity analysis call for the differentiation of the state system:

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 & \Omega \times (0, T), \\ \mathbf{U}(x, 0) = \mathbf{g}(x) & \Omega, \end{cases}$$

[8] Bardos, C., Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

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For the **Burgers' equation**:

$$\mathbf{F}(\mathbf{U}) = \frac{u^2}{2} \qquad \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = u u_a$$

This can be done under **hypotheses of regularity** of the state ***U*** [8].

If these techniques are applied to hyperbolic equations, **Dirac delta functions** will appear in the sensitivity.

[8] Bardos, C., Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

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In order to have a sensitivity system which is valid also when the state is discontinuous, we add a correction term [9]:

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S} & \Omega \times (0, T), \\ \mathbf{U}_a(x, 0) = \mathbf{g}_a(x) & \Omega, \end{cases}$$

defined as follows:

$$\mathbf{S} = \sum_{k=1}^{N_s} \boldsymbol{\rho}_k \delta(x - x_{k,s}(t)),$$

N_s: number of discontinuities

$x_{k,s}$: position of the k-th discontinuity

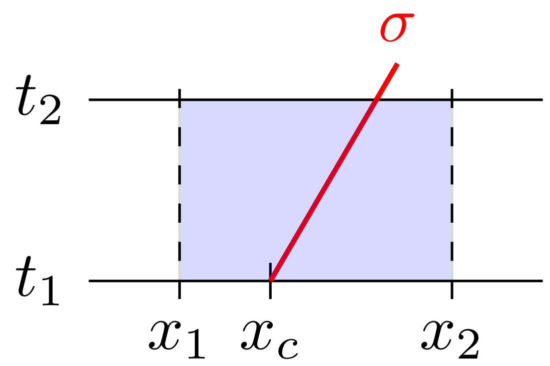
$\boldsymbol{\rho}_k$: amplitude of the k-th correction (to be computed)

Remark: a **shock detector** is necessary to discretise such source term.

[9] Guinot, V., Delenne, C., Cappelaere, B. (2009). An approximate Riemann solver for sensitivity equations with discontinuous solutions. *Advances in Water Resources*, *32*(1), 61-77.

Definition of the source term

To compute the amplitude of the correction, we consider an infinitesimal control volume containing a single discontinuity:



By integrating the sensitivity equations with the source term on the control volume, one has:

$$\boldsymbol{\rho}_k = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + \mathbf{F}_a^+ - \mathbf{F}_a^-$$

Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma_k = \mathbf{F}^+ - \mathbf{F}^-$

Differentiating them w.r.t. the parameter:

$$(\mathbf{U}_a^+ - \mathbf{U}_a^-)\sigma_k + (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x\mathbf{U}^+ - \partial_x\mathbf{U}^-)\partial_a x_{k,s}(t) =$$
$$= \mathbf{F}_a^+ - \mathbf{F}_a^- + \left(\frac{\partial\mathbf{F}(\mathbf{U}^+)}{\partial\mathbf{U}}\partial_x\mathbf{U}^+ - \frac{\partial\mathbf{F}(\mathbf{U}^-)}{\partial\mathbf{U}}\partial_x\mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

Finally, we obtain the following amplitude:

$$\boldsymbol{\rho}_k = (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x\mathbf{U}^+ - \partial_x\mathbf{U}^-)\partial_a x_{k,s}(t) - \left(\frac{\partial\mathbf{F}(\mathbf{U}^+)}{\partial\mathbf{U}}\partial_x\mathbf{U}^+ - \frac{\partial\mathbf{F}(\mathbf{U}^-)}{\partial\mathbf{U}}\partial_x\mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

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The Euler equations are:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = 0, \end{cases}$$

Eigenvalues:

$$\lambda_1(\mathbf{U}) = u - c, \quad \lambda_2(\mathbf{U}) = u, \quad \lambda_3(\mathbf{U}) = u + c.$$

Eigenvectors:

$$\mathbf{r}_1(\mathbf{U}) = (1, u - c, H - uc)^t, \quad \mathbf{r}_2(\mathbf{U}) = (1, u, \tfrac{u^2}{2})^t, \quad \mathbf{r}_3(\mathbf{U}) = (1, u + c, H + uc)^t.$$

The Riemann problem for the Euler equations

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Genuinely nonlinear

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Linearly degenerate

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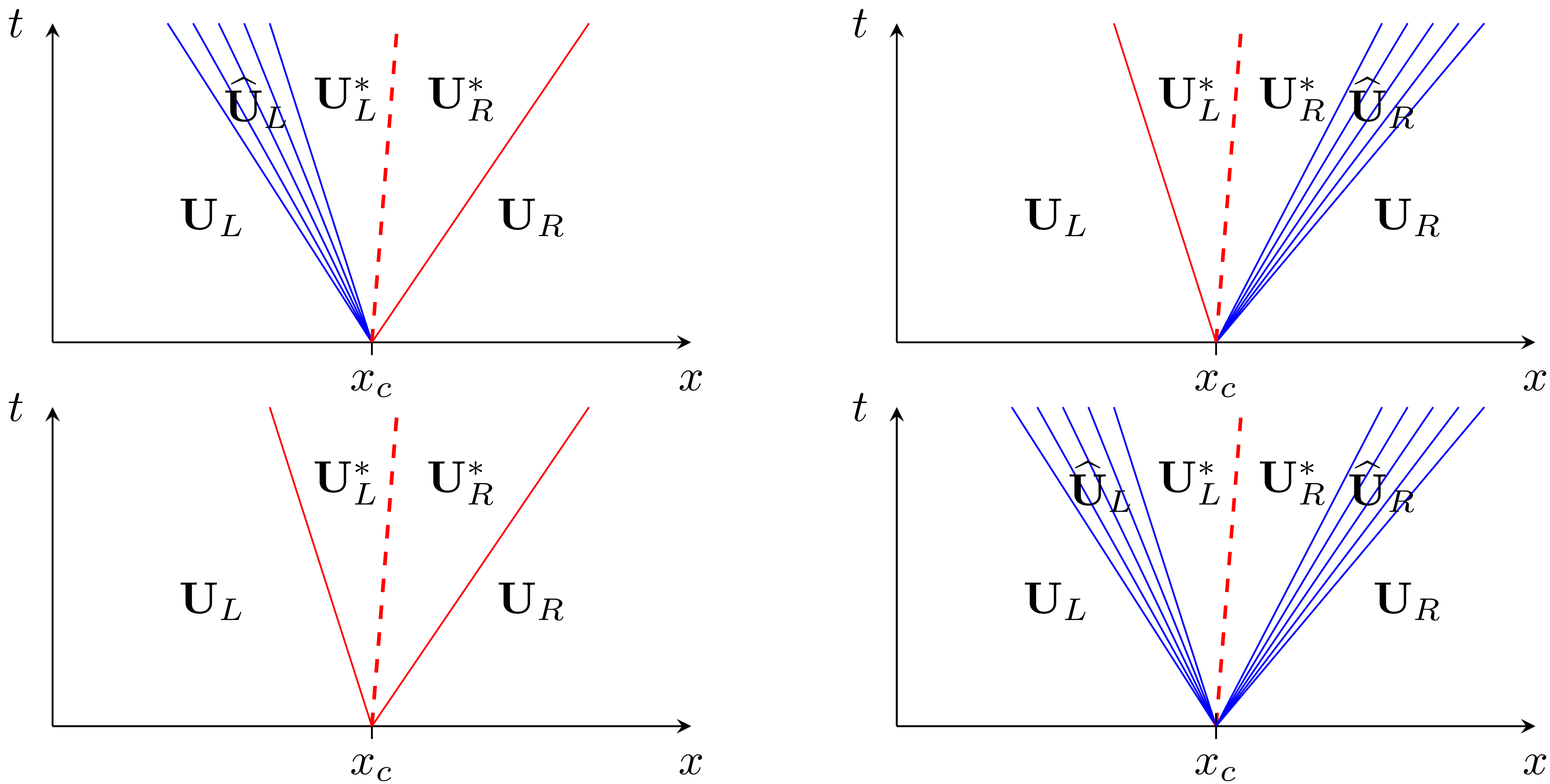

The sensitivity system is:

$$\begin{cases} \partial_t \rho_a + \partial_x (\rho u)_a = S_1, \\ \partial_t (\rho u)_a + \partial_x (\rho_a u^2 + 2\rho u u_a + p_a) = S_2, \\ \partial_t (\rho E)_a + \partial_x (u_a(\rho E + p) + u((\rho E)_a + p_a)) = S_3, \end{cases}$$

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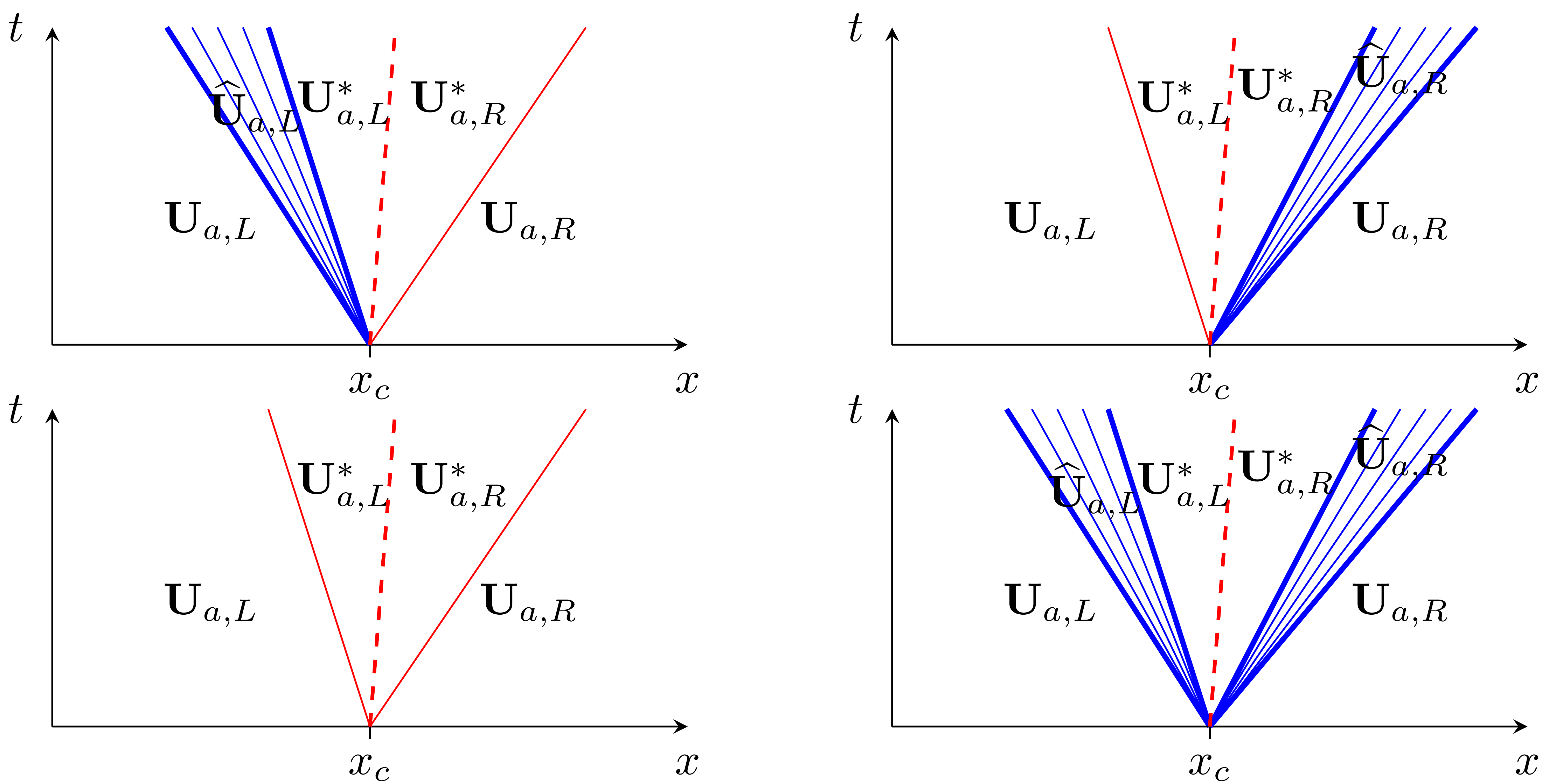

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Godunov-type schemes

approximate Riemann solvers are used

Step 0 : initial data discretisation

Step 1 : solution of a Riemann problem for each interface $x_{j-1/2}$ obtaining $\mathbf{v}(x, t^{n+1})$

Step 2 : average $\mathbf{V}_j^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, t^{n+1}) \mathrm{d}x$

Remark: the state and the sensitivity systems are not solved as a global system.

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

Remark: HLL-type schemes cannot be used for the state, two intermediate star states are necessary to have a well-defined the source term for the sensitivity.

Approximate Riemann solver for the state

- First order Roe-type scheme

$$\lambda_1^{ROE} = \tilde{u} - \tilde{c} \qquad \lambda_2^{ROE} = \tilde{u} \qquad \lambda_3^{ROE} = \tilde{u} + \tilde{c} \qquad \text{Roe-averaged eigenvalues}$$

$$\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \alpha_i \tilde{\mathbf{r}}_i \qquad \text{decomposition along Roe-averaged eigenvectors}$$

$$\mathbf{U}_L^* = \mathbf{U}_L + \alpha_1 \tilde{\mathbf{r}}_1 \qquad \mathbf{U}_R^* = \mathbf{U}_R - \alpha_3 \tilde{\mathbf{r}}_3$$

Classical numerical schemes

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

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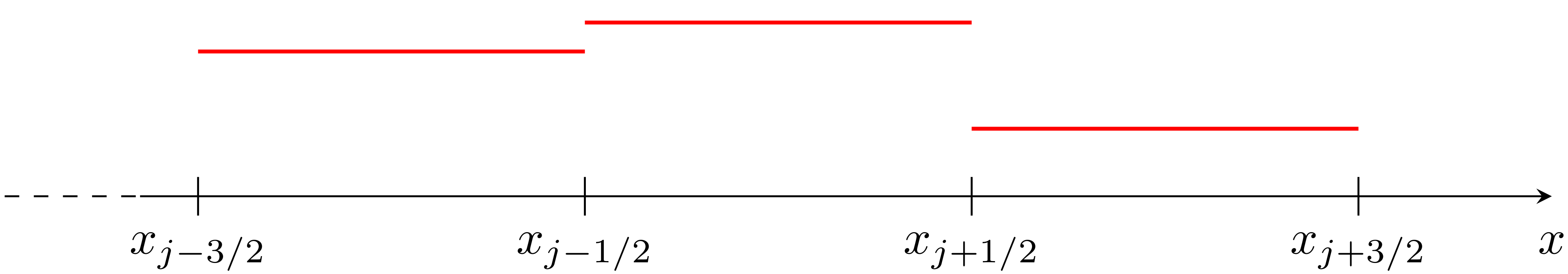
Approximate Riemann solver for the state

- First order Roe-type scheme
- Second order Roe-type scheme

  Time discretisation: two-step Runge-Kutta method

  Space discretisation: MUSCL-type scheme [10]



[10] Bouchut, F. (2004). Nonlinear stability of finite Volume Methods for hyperbolic conservation laws: And Well-Balanced schemes for sources. Springer Science & Business Media.

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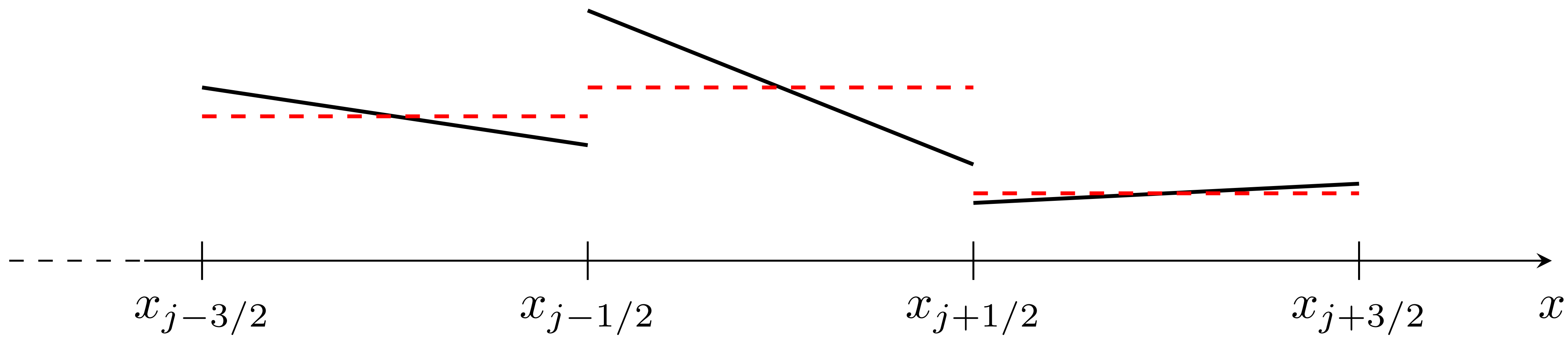

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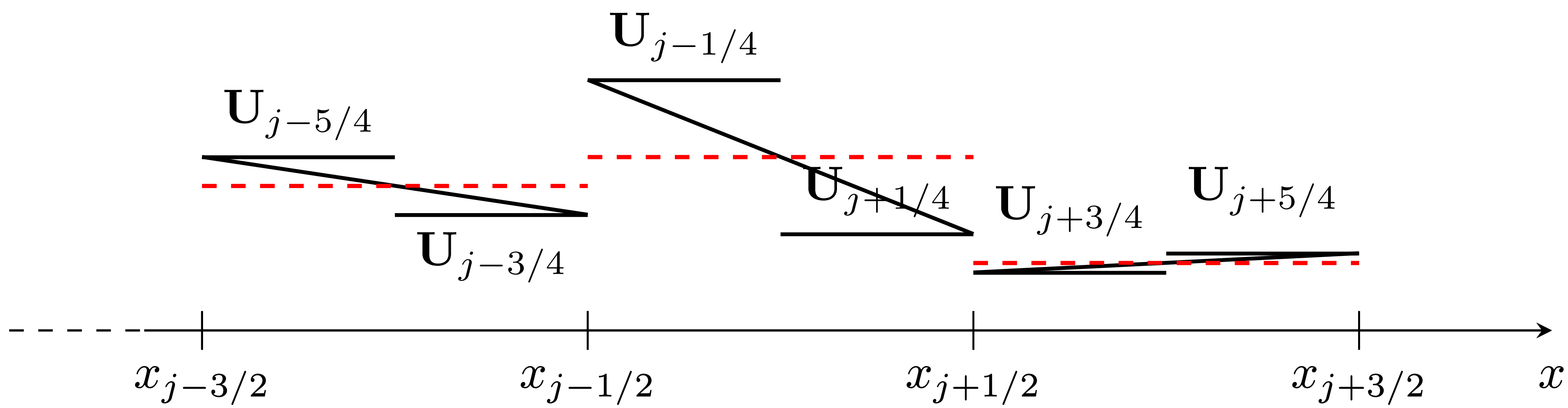

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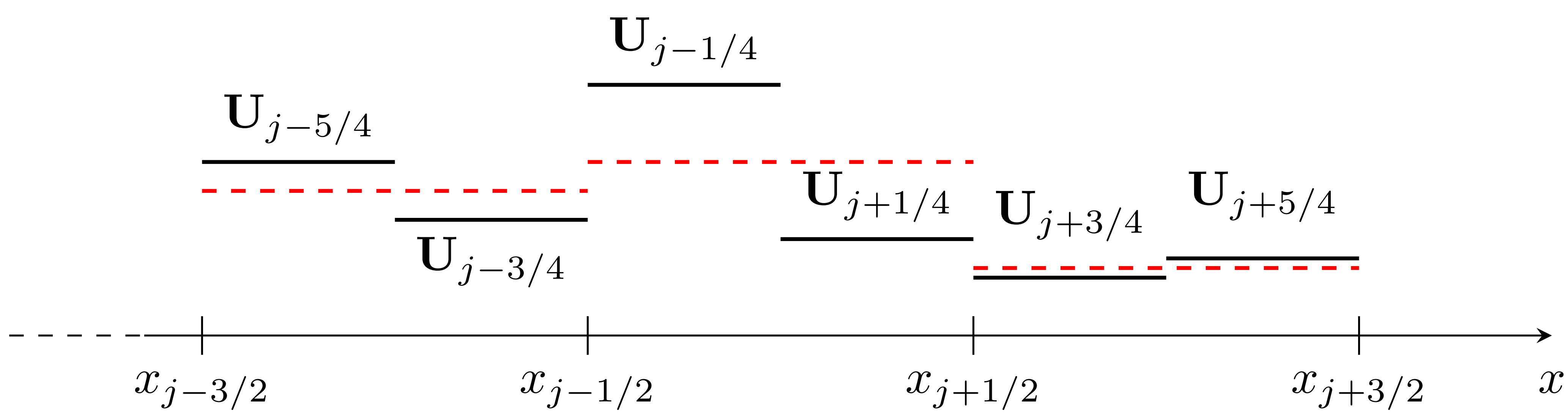

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Approximate Riemann solvers for the sensitivity

- HLL-type scheme: simpler structure than the state solver.

HLL consistency conditions yield:

$$\mathbf{U}^*_{a,j-1/2} = \frac{1}{\lambda_3^{ROE} - \lambda_1^{ROE}} \Big(\lambda_3^{ROE} \mathbf{U}^n_{a,j} - \lambda_1^{ROE} \mathbf{U}^n_{a,j-1} - \mathbf{F}_a(\mathbf{U}_j, \mathbf{U}_{a,j}) + \mathbf{F}_a(\mathbf{U}_{j-1}, \mathbf{U}_{a,j-1}) + \mathbf{S}_{j-1/2} \Big)$$

$$\begin{aligned} \mathbf{S}_{j-1/2} =& \partial_a \lambda^{ROE}_{1,j-1/2} (\mathbf{U}^*_{L,j-1/2} - \mathbf{U}_{j-1}) d_{1,j-1/2} \\ &+ \partial_a \lambda^{ROE}_{2,j-1/2} (\mathbf{U}^*_{R,j-1/2} - \mathbf{U}^*_{L,j-1/2}) \\ &+ \partial_a \lambda^{ROE}_{3,j-1/2} (\mathbf{U}_j - \mathbf{U}^*_{R,j-1/2}) d_{3,j-1/2} \end{aligned}$$

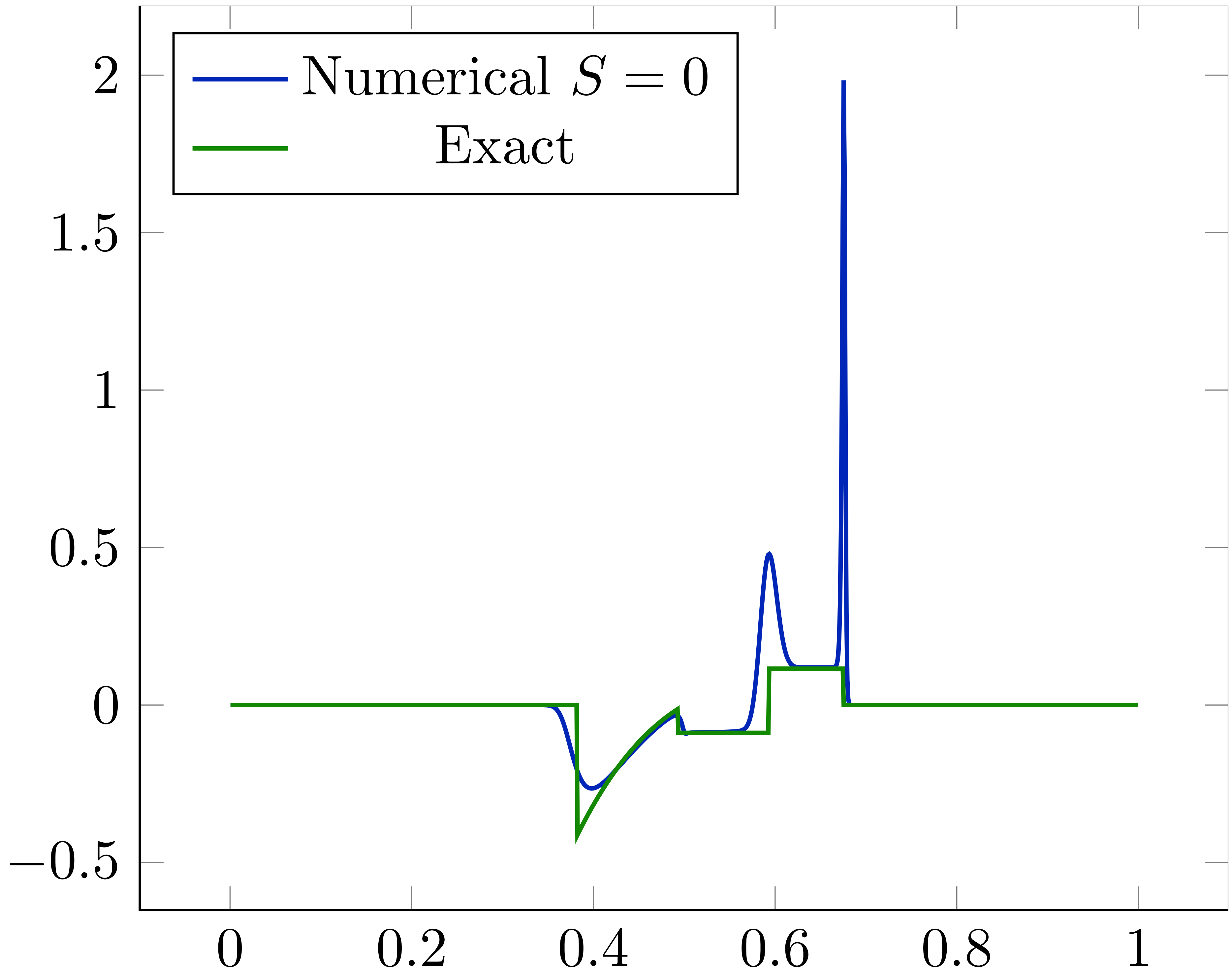
- HLLC-type scheme: same structure as the state.

HLL consistency conditions + Rankine-Hugoniot conditions. Equivalent to:

$$\mathbf{U}^*_{a,L} = \mathbf{U}_{a,L} + \alpha_{1,a} \tilde{\mathbf{r}}_1 + \alpha_1 \tilde{\mathbf{r}}_{1,a} \qquad \mathbf{U}^*_{a,R} = \mathbf{U}_{a,R} - \alpha_{3,a} \tilde{\mathbf{r}}_3 - \alpha_3 \tilde{\mathbf{r}}_{3,a}$$

Classical numerical schemes

$\rho_{p_L}(x,T)$



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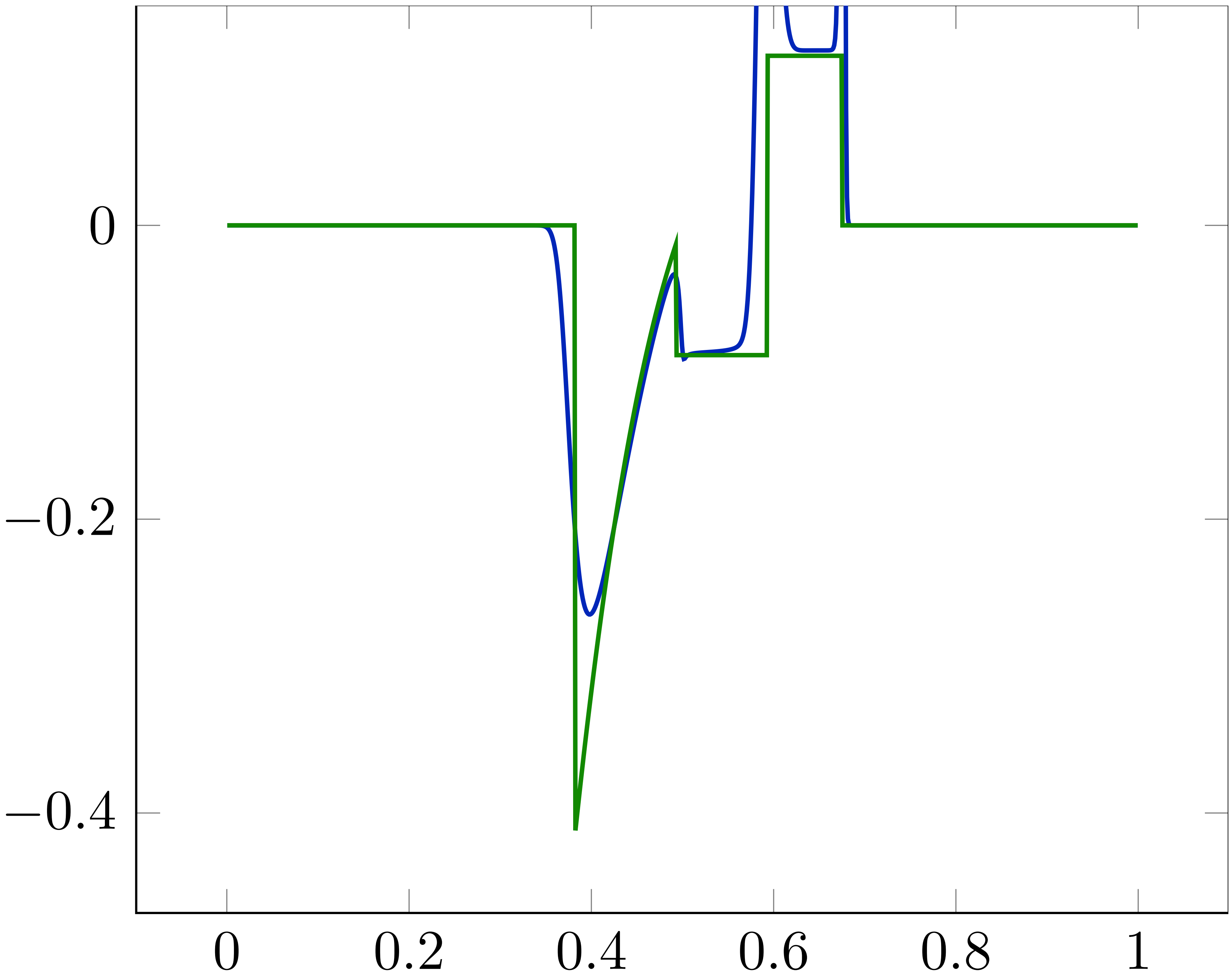

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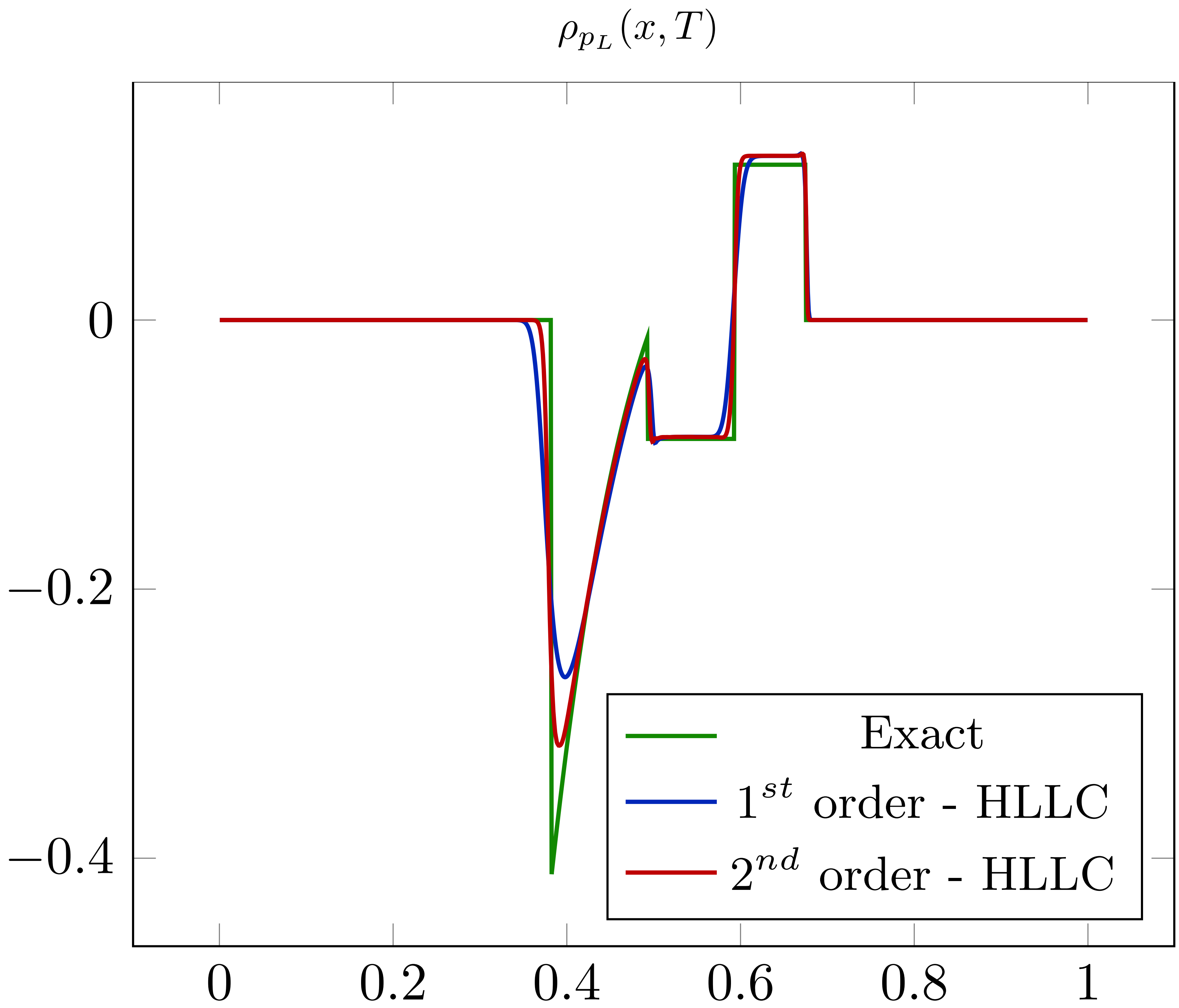
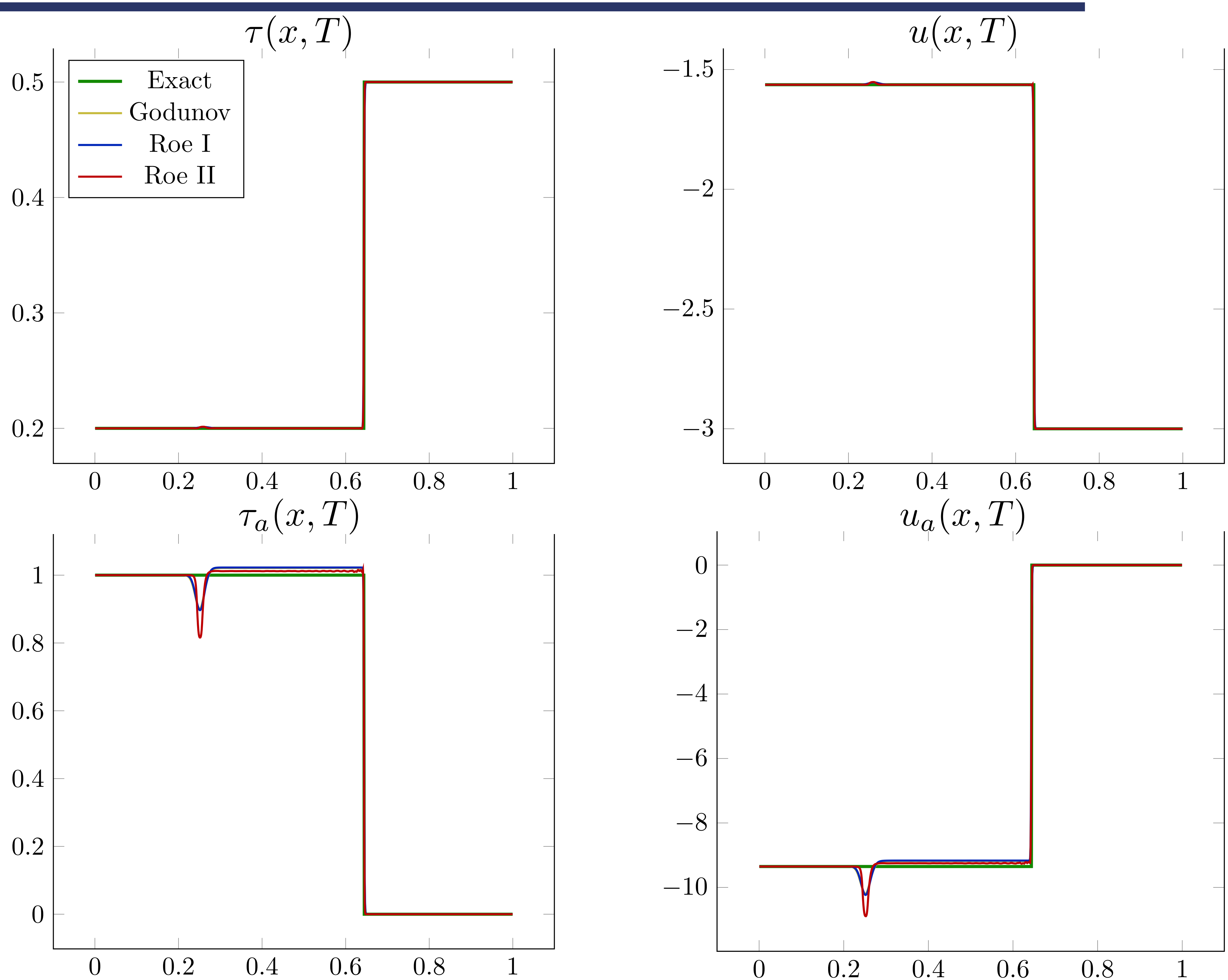


Isolated shock for the p-system



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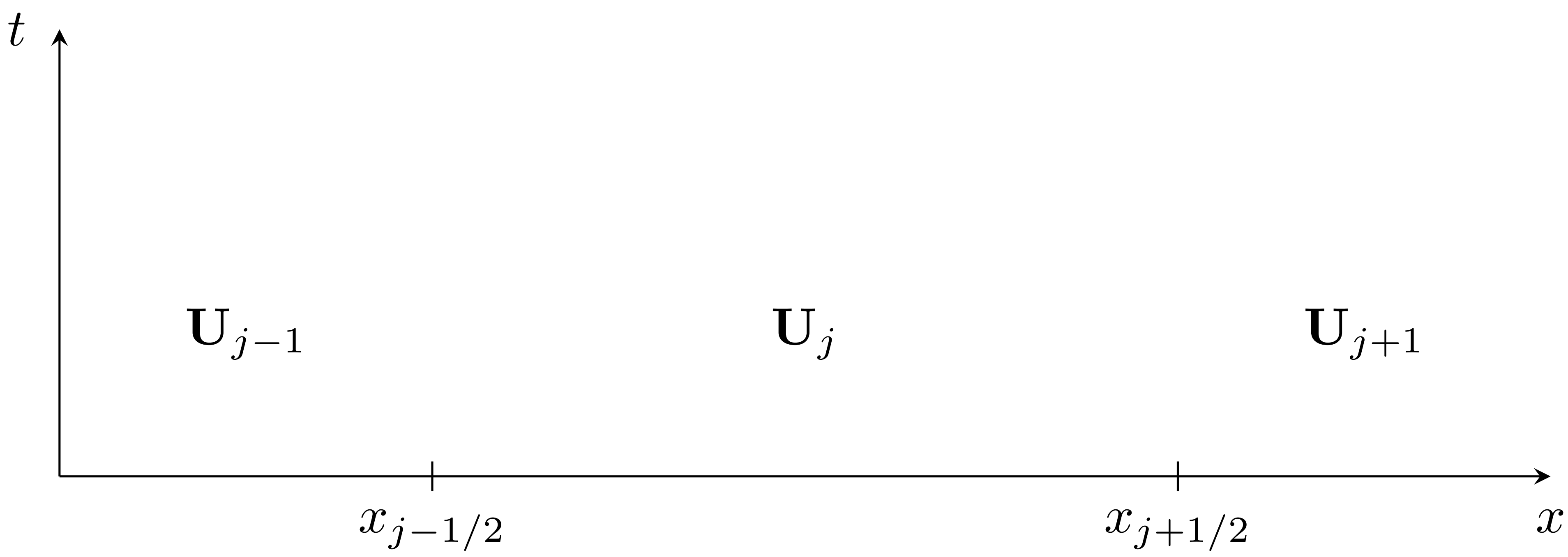
Scheme without numerical diffusion

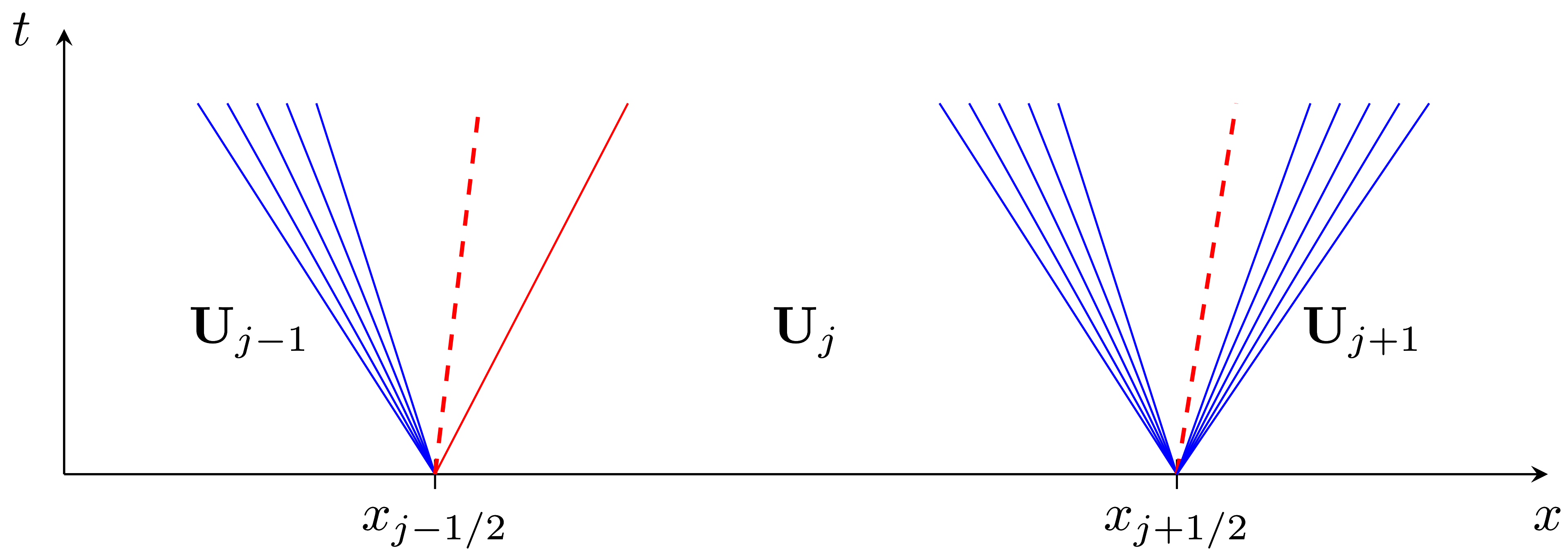
Step 0 : initial data discretisation



Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

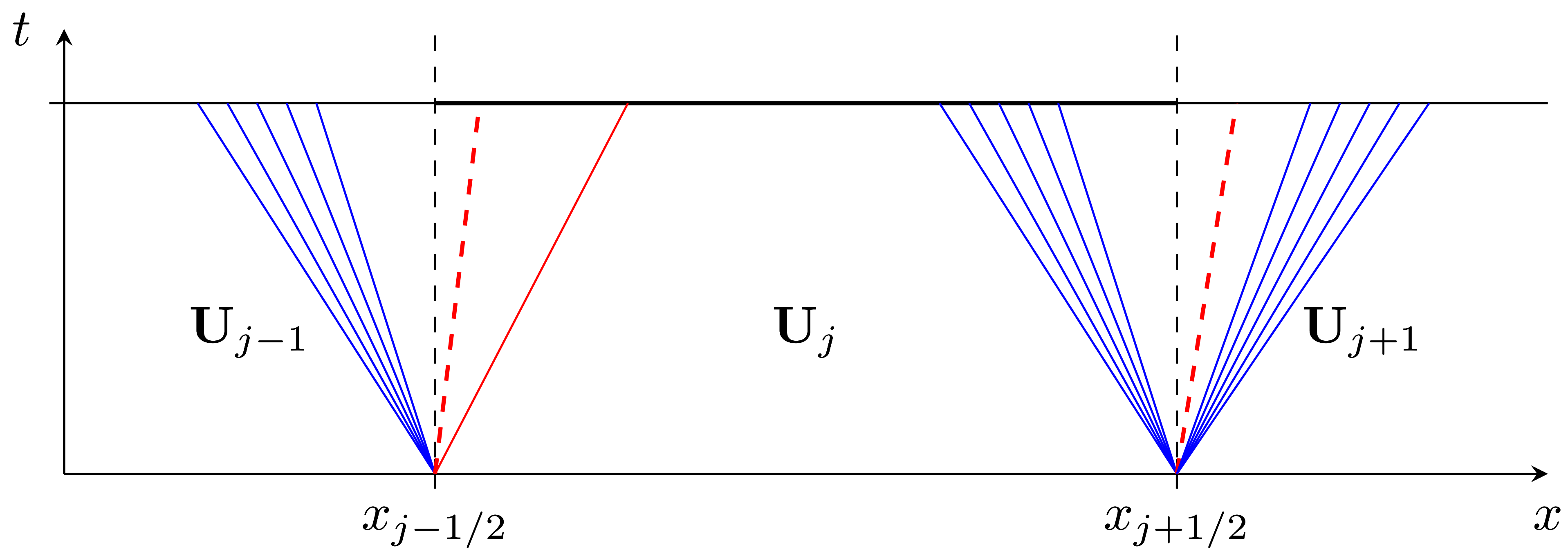


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : average



Scheme without numerical diffusion

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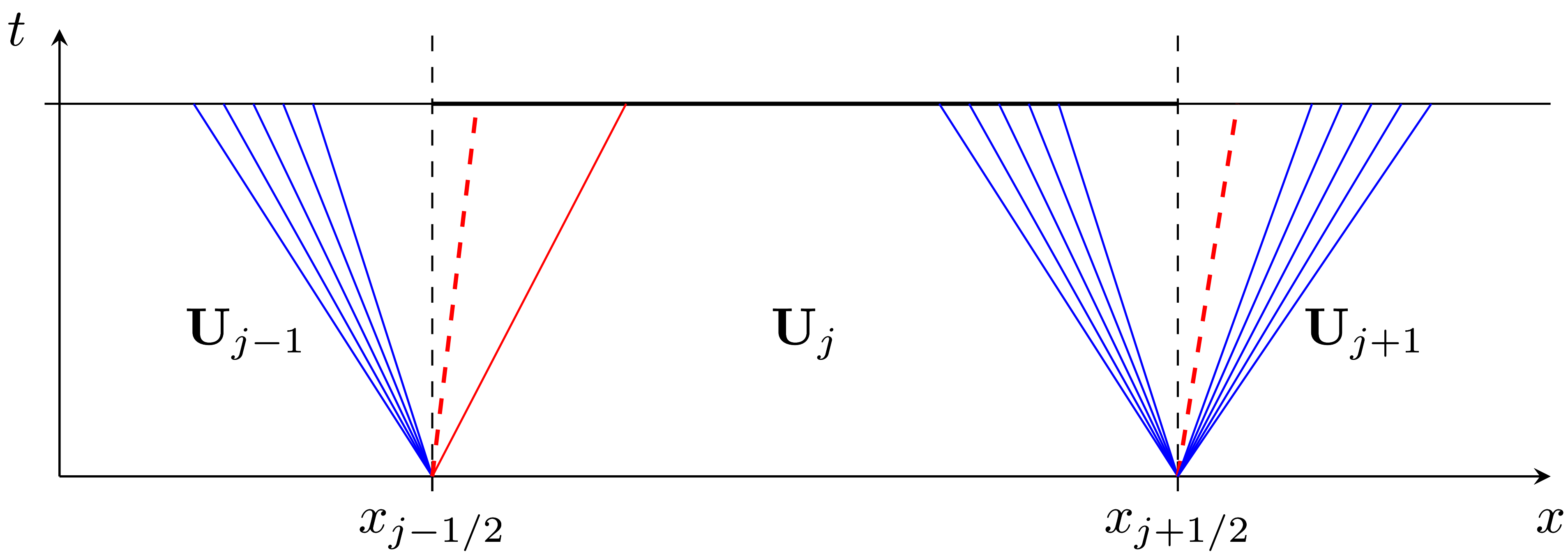
Step 2 : ~~average~~

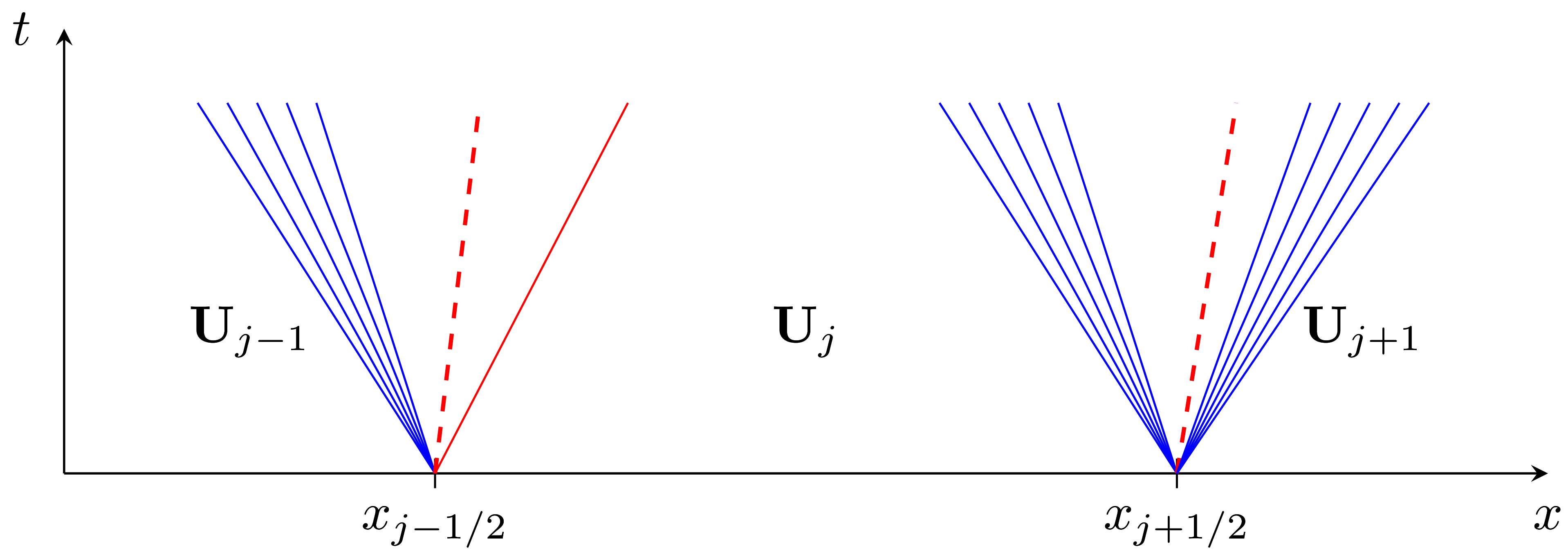


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed [11]



[11] Chalons, C., Goatin, P. (2008). Godunov scheme and sampling technique for computing phase transitions in traffic flow modeling. *Interfaces and Free Boundaries*, *10*(2), 197-221.

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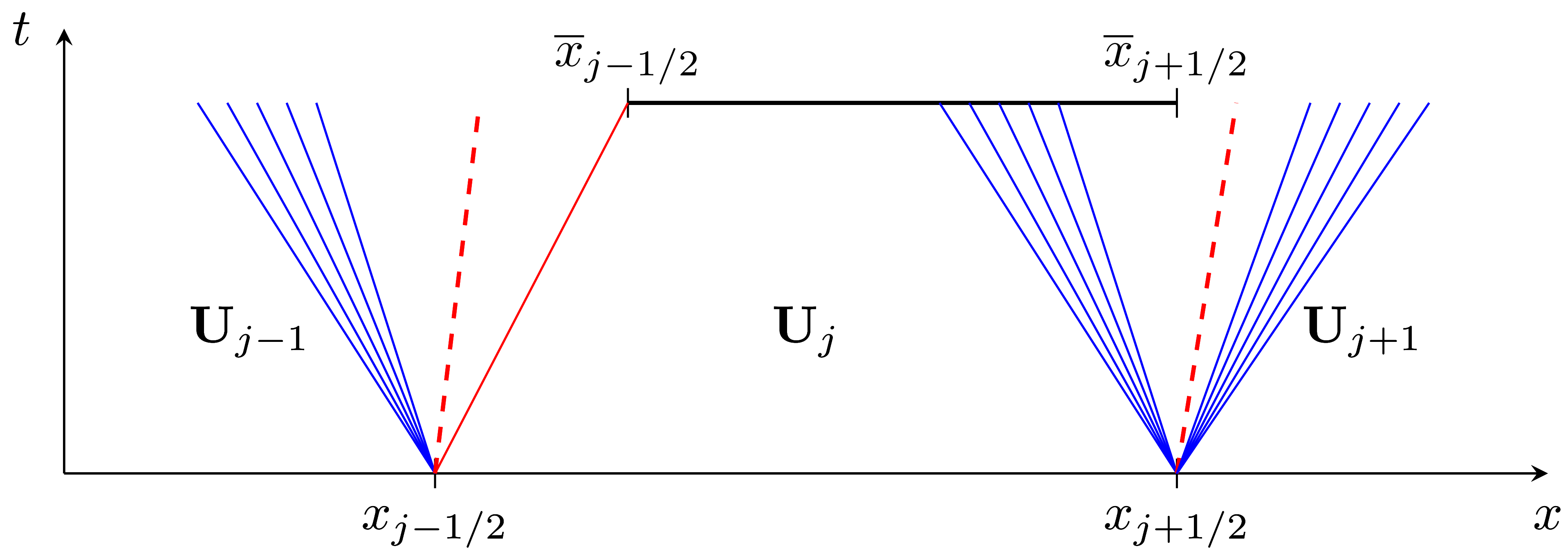


$$\overline{x}_{j-1/2} = x_{j-1/2} + \sigma_{j-1/2}\Delta t$$

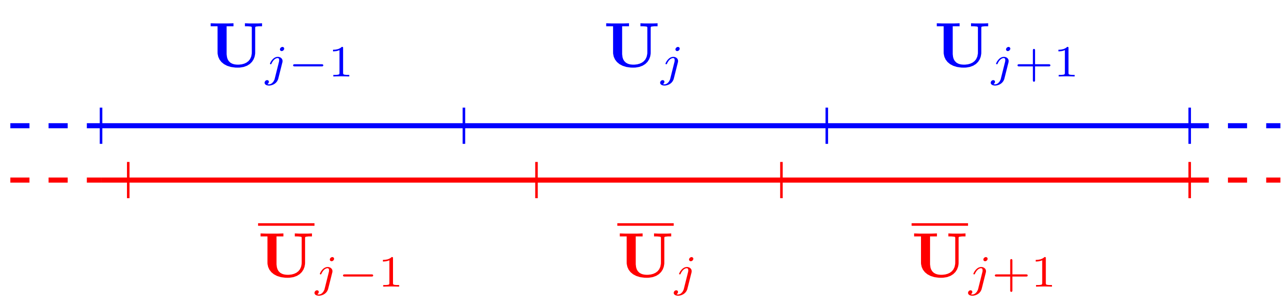
[11] Chalons, C., Goatin, P. (2008). Godunov scheme and sampling technique for computing phase transitions in traffic flow modeling. *Interfaces and Free Boundaries*, *10*(2), 197-221.

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed

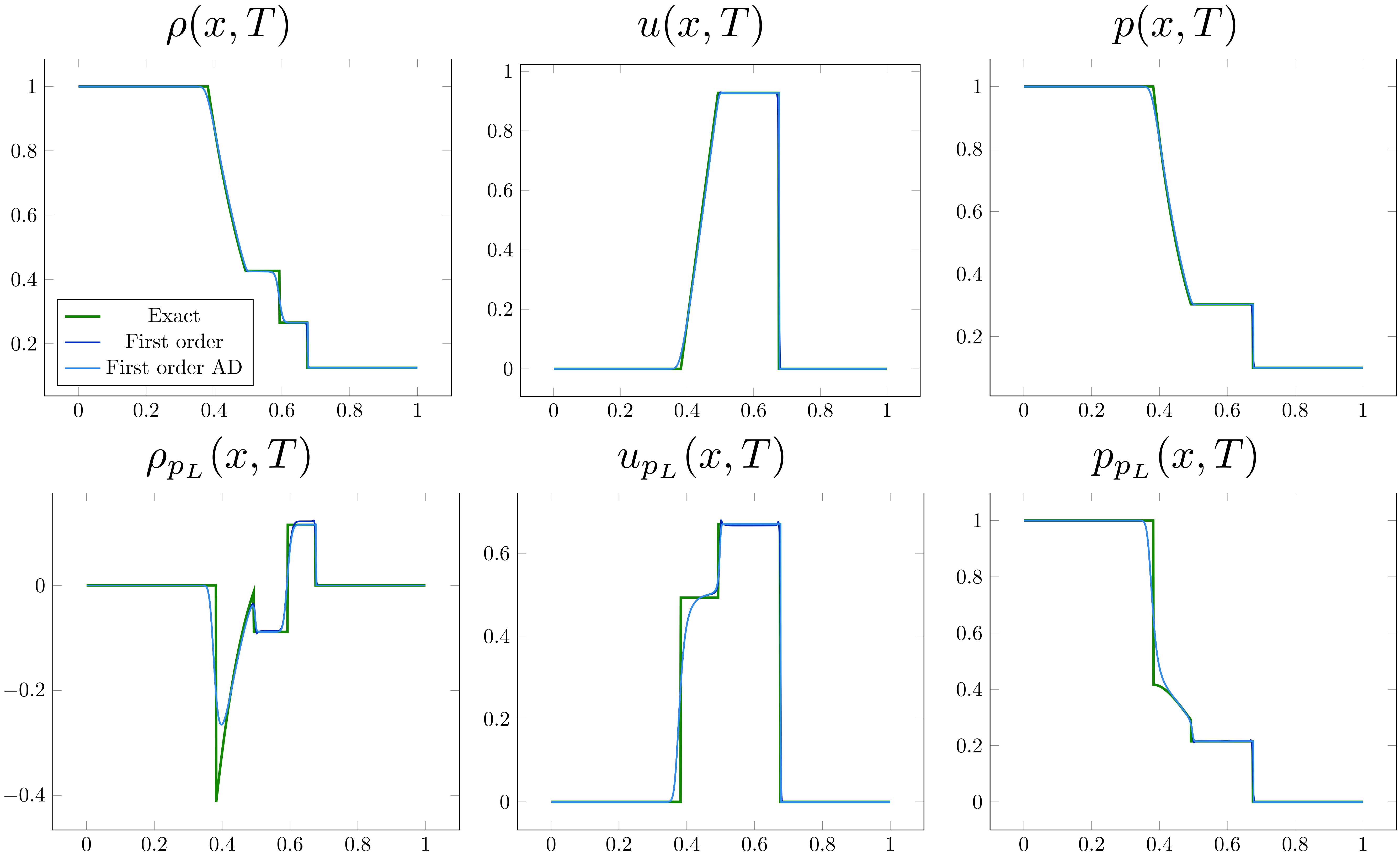
Step 3 : projection on the initial mesh [12]



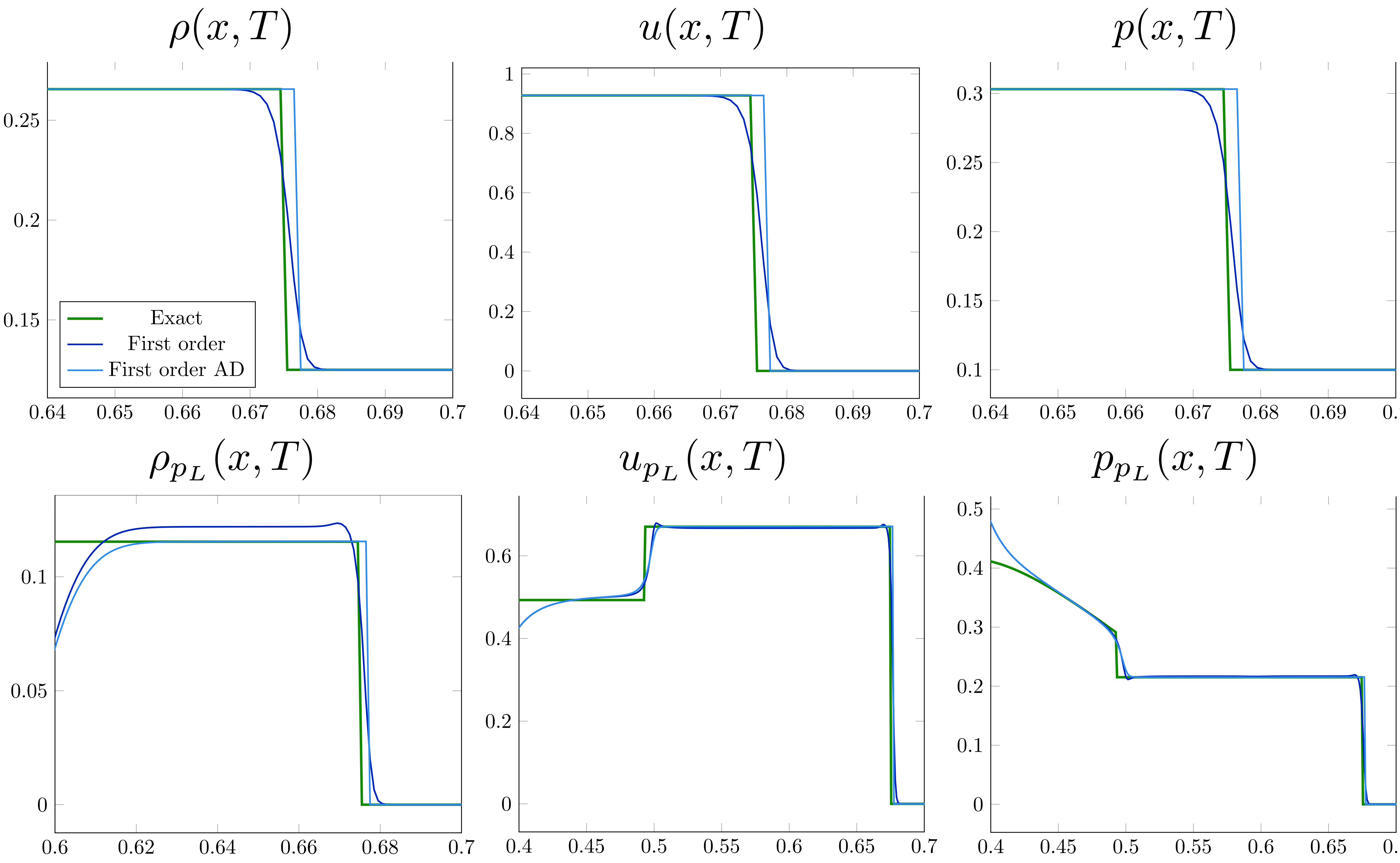
$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } \alpha \in \left(0, \frac{\Delta t}{\Delta x}\max(\sigma_{j-1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j} & \text{if } \alpha \in \left[\frac{\Delta t}{\Delta x}\max(\sigma_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x}\min(\sigma_{j+1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } \alpha \in \left[1 + \frac{\Delta t}{\Delta x}\min(\sigma_{j+1/2}, 0), 1\right). \end{cases}$$

$$\alpha \sim \mathcal{U}([0,1])$$

[12] Glimm, J. (1965). Solutions in the large for nonlinear hyperbolic systems of equations. *Communications on pure and applied mathematics*, *18*(4), 697-715.

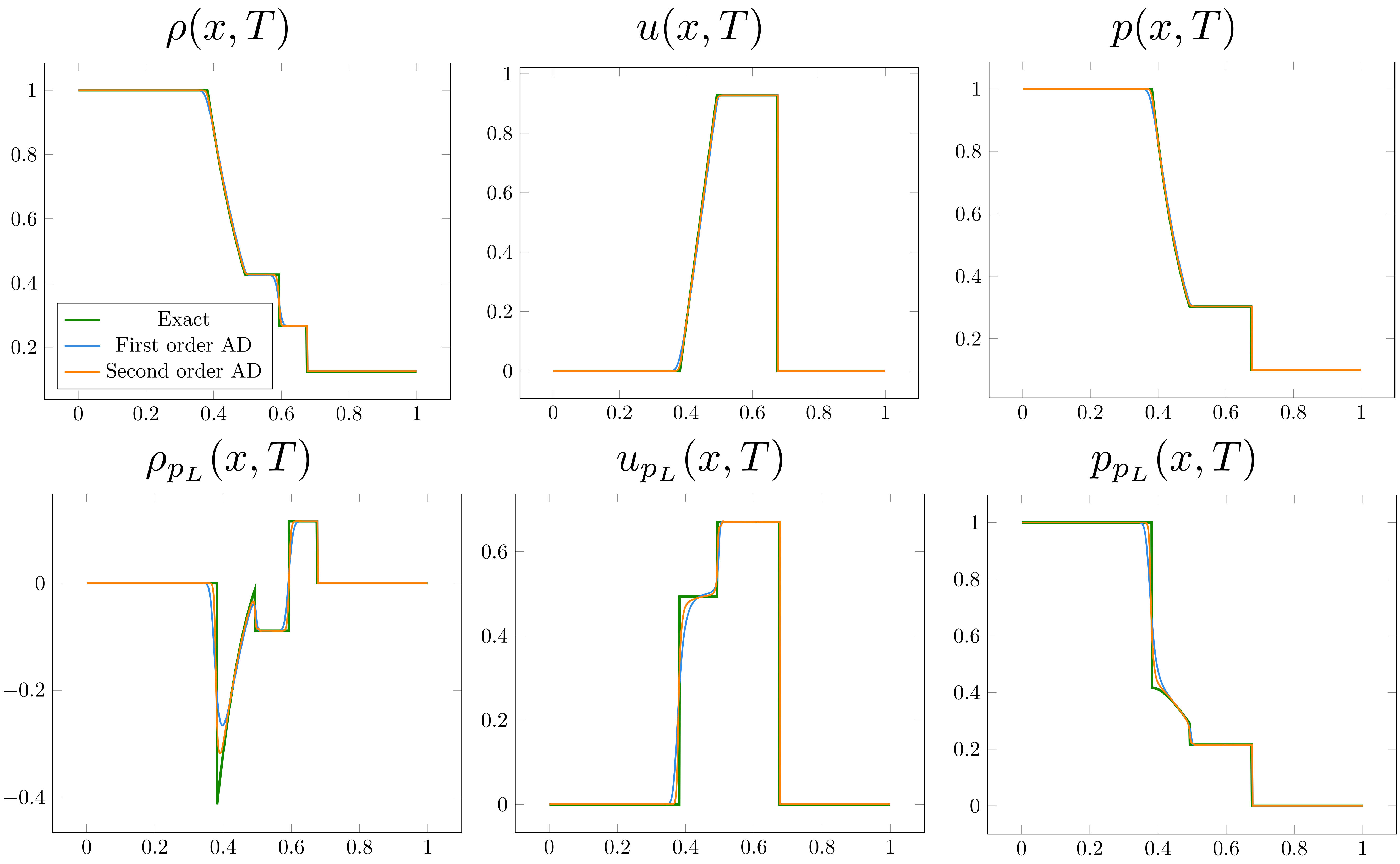


$\rho(x,T)$
$u(x,T)$
$p(x,T)$
$\rho_{p_L}(x,T)$
$u_{p_L}(x,T)$
$p_{p_L}(x,T)$
Exact
First order
First order AD
0
0.2
0.4
0.6
0.8
1
−0.2
−0.4

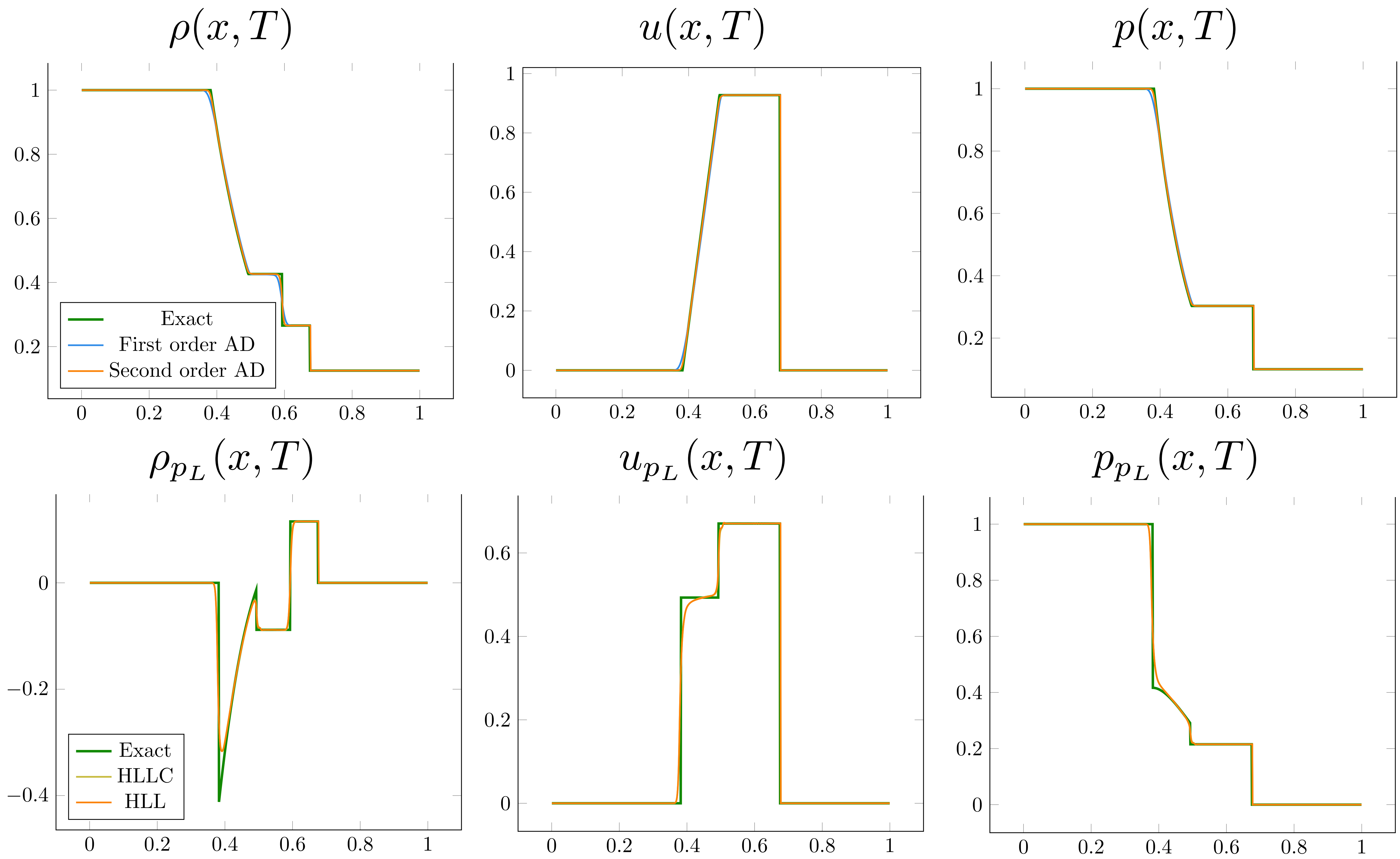


$\rho(x,T)$
$u(x,T)$
$p(x,T)$
Exact
First order
First order AD
$\rho_{p_L}(x,T)$
$u_{p_L}(x,T)$
$p_{p_L}(x,T)$

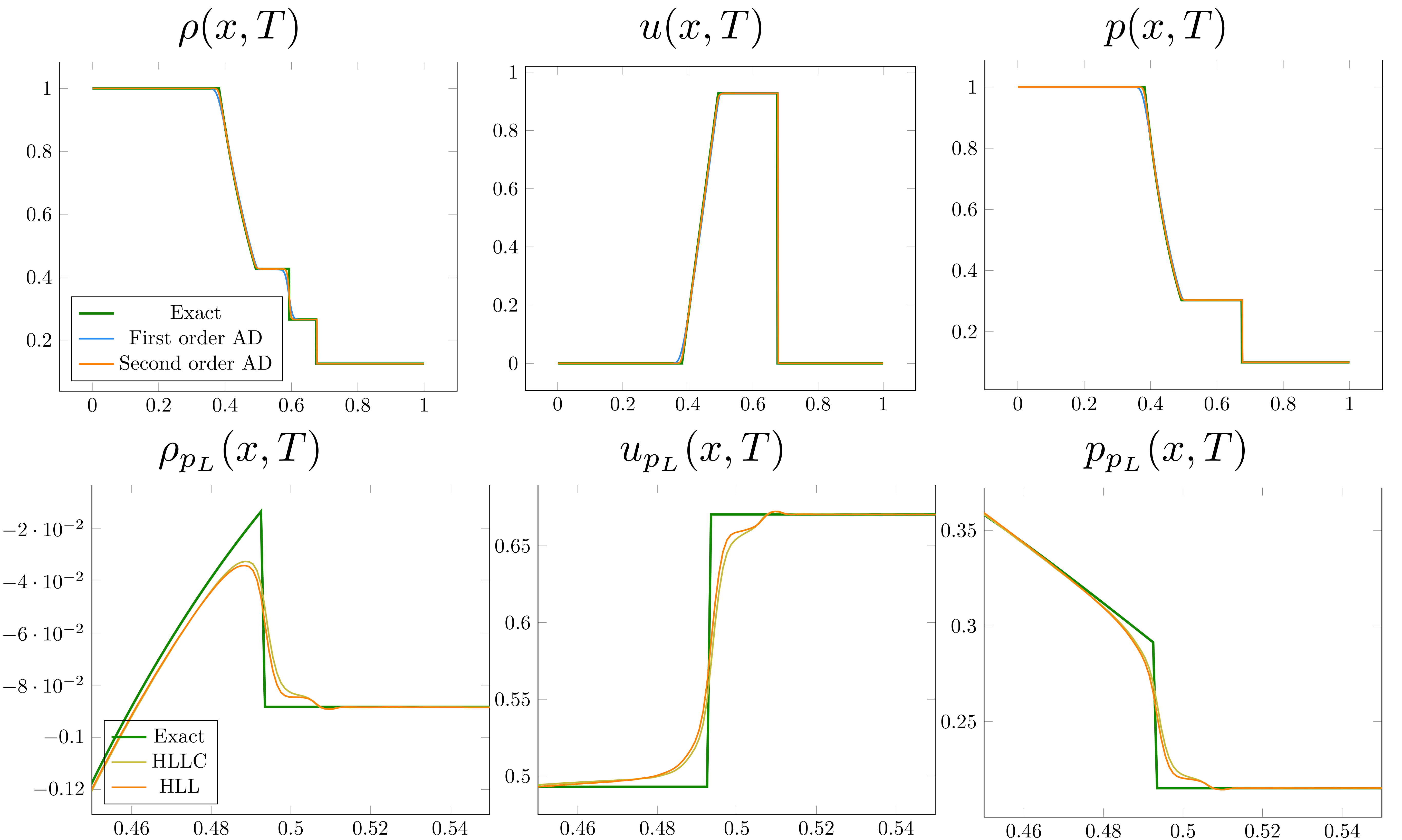
Numerical results

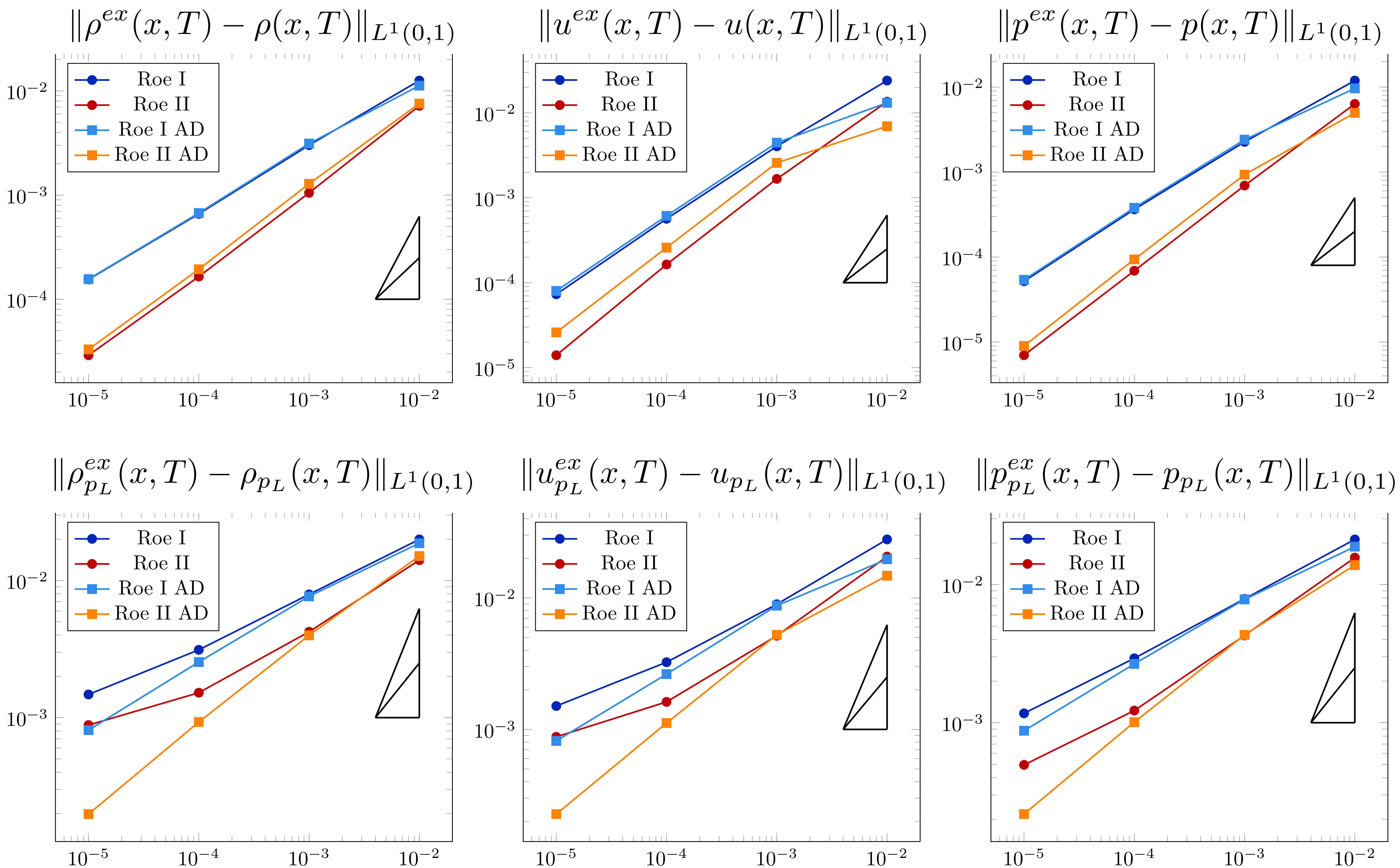


Numerical results



Numerical results





$\|\rho^{ex}(x,T) - \rho(x,T)\|_{L^1(0,1)}$
$\|u^{ex}(x,T) - u(x,T)\|_{L^1(0,1)}$
$\|p^{ex}(x,T) - p(x,T)\|_{L^1(0,1)}$
$\|\rho^{ex}_{p_L}(x,T) - \rho_{p_L}(x,T)\|_{L^1(0,1)}$
$\|u^{ex}_{p_L}(x,T) - u_{p_L}(x,T)\|_{L^1(0,1)}$
$\|p^{ex}_{p_L}(x,T) - p_{p_L}(x,T)\|_{L^1(0,1)}$
Roe I
Roe II
Roe I AD
Roe II AD
10^{-5}
10^{-4}
10^{-3}
10^{-2}

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Riemann problem for the Euler equations and their sensitivity
- Classical numerical schemes
- Anti-diffusive numerical schemes
- **Applications**

Uncertainty propagation

Let **a** be a random vector, with the following average and variance:

$$\mu_{\mathbf{a}} = \begin{bmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_M} \end{bmatrix}, \qquad \sigma_{\mathbf{a}} = \begin{bmatrix} \sigma^2_{a_1} & \mathrm{cov}(a_1, a_2) & \dots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_1, a_2) & \sigma^2_{a_2} & \dots & \mathrm{cov}(a_2, a_M) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_1, a_M) & \dots & & \sigma^2_{a_M} \end{bmatrix},$$

Aim: determine a **confidence interval** $CI_X = [\mu_X - \kappa\sigma_X, \mu_X + \kappa\sigma_X]$

Monte Carlo approach: *N* samples of the state X_k

$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma^2_X = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

Uncertainty propagation

Sensitivity approach: let us consider the following first order Taylor expansion

$$X(\mathbf{a}) = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M}(a_i - \mu_{a_i})X_{a_i}(\mu_{\mathbf{a}}) + o(\|\mathbf{a} - \mu_{\mathbf{a}}\|^2)$$

Then, computing, the average one has:

$$\mu_X = E[X(\mathbf{a})] = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})E[a_i - \mu_{a_i}] = X(\mu_{\mathbf{a}}),$$

And for the variance:

$$\sigma_X^2 = E[(X(\mathbf{a}) - \mu_X)^2] = E\left[\left(\sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})(a_i - \mu_{a_i})\right)^2\right] =$$

$$= \sum_{i=1}^{M} X_{a_i}^2(\mu_{\mathbf{a}})E[(a_i - \mu_{a_i})^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i}(\mu_{\mathbf{a}})X_{a_j}(\mu_{\mathbf{a}})E[(a_i - \mu_{a_i})(a_j - \mu_{a_j})].$$

Therefore, we have the following first order estimates:

$$\mu_X = X(\mu_{\mathbf{a}}), \quad \sigma_X^2 = \sum_{i=1}^{M} X_{a_i}^2 \sigma_{a_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i} X_{a_j} \operatorname{cov}(a_i, a_j).$$

Test case:

Riemann problem with uncertain parameters: $\mathbf{a} = (\rho_L, \rho_R, u_L, u_R, p_L, p_R)^t$

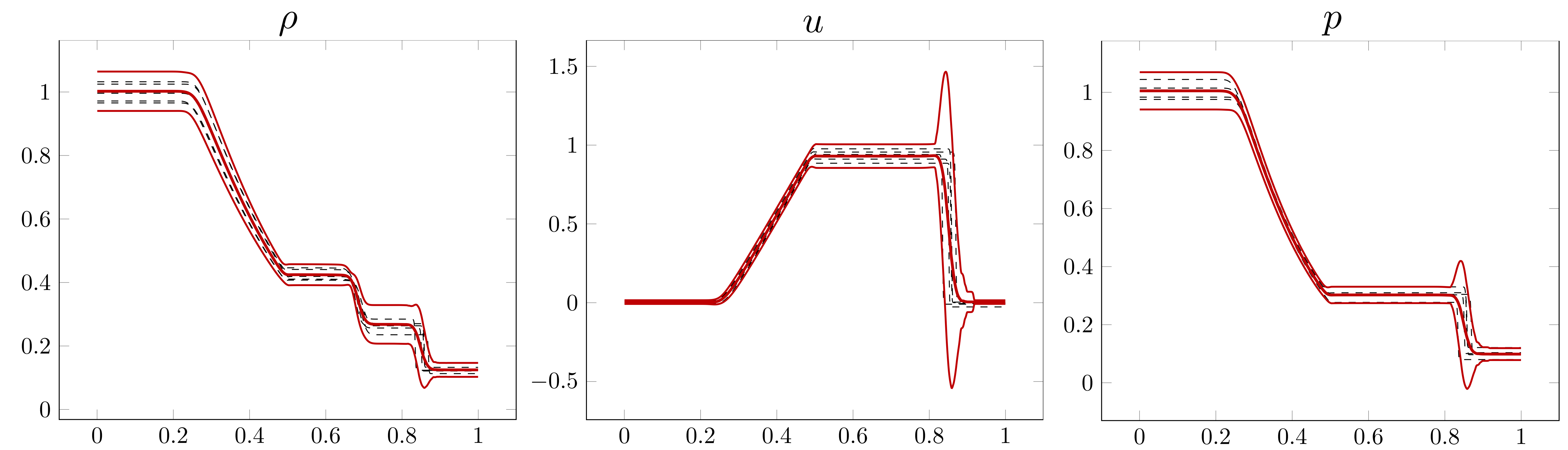
with the following average and covariance matrix:

$$\mu_{\mathbf{a}} = (1, 0.125, 0, 0, 1, 0.1)^t, \quad \sigma_{\mathbf{a}} = \mathrm{diag}(0.001, 0.000125, 0.0001, 0.0001, 0.001, 0.0001).$$

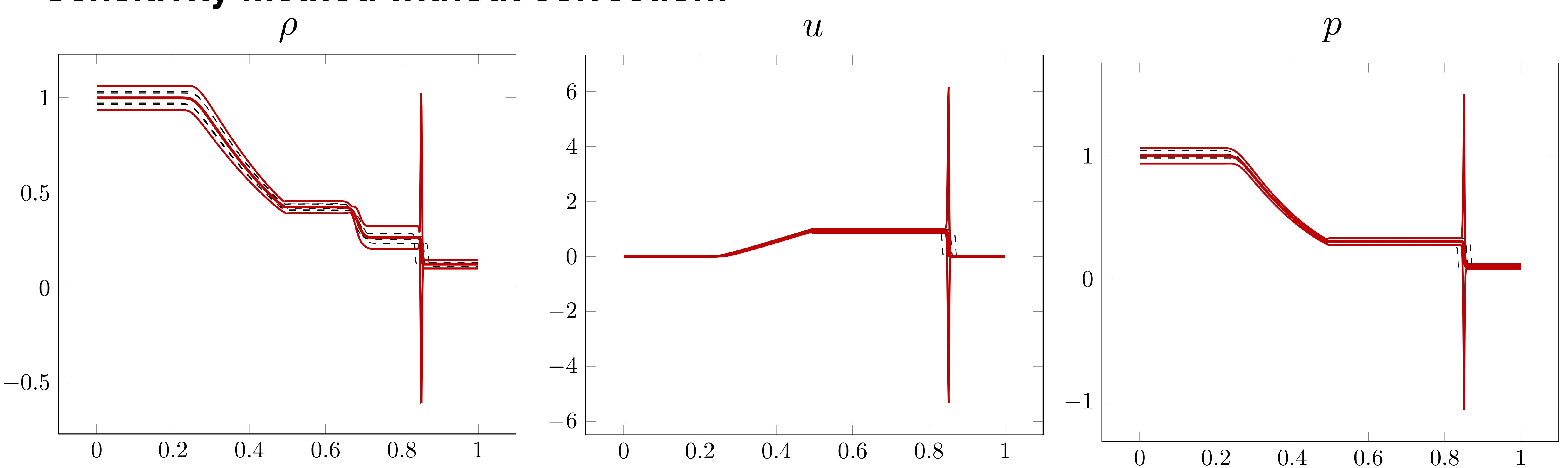
Since the covariance matrix is diagonal, the previous estimate is simplified:

$$\sigma_X^2 = \sum_{i=1}^{6} X_{a_i}^2 \sigma_{a_i}^2$$

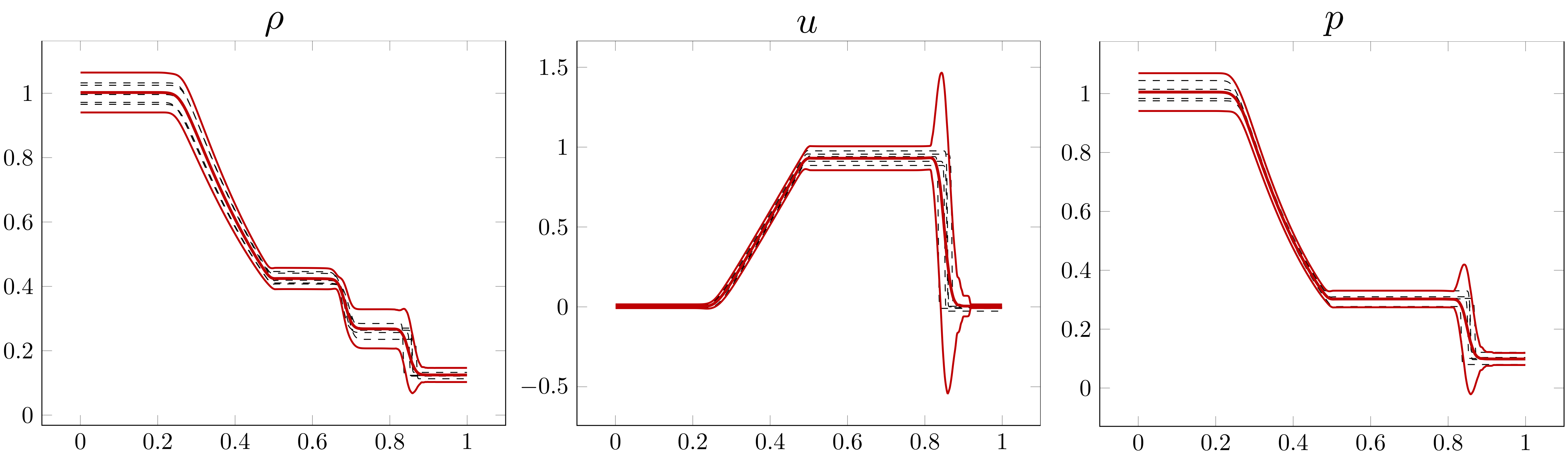
Monte Carlo method:



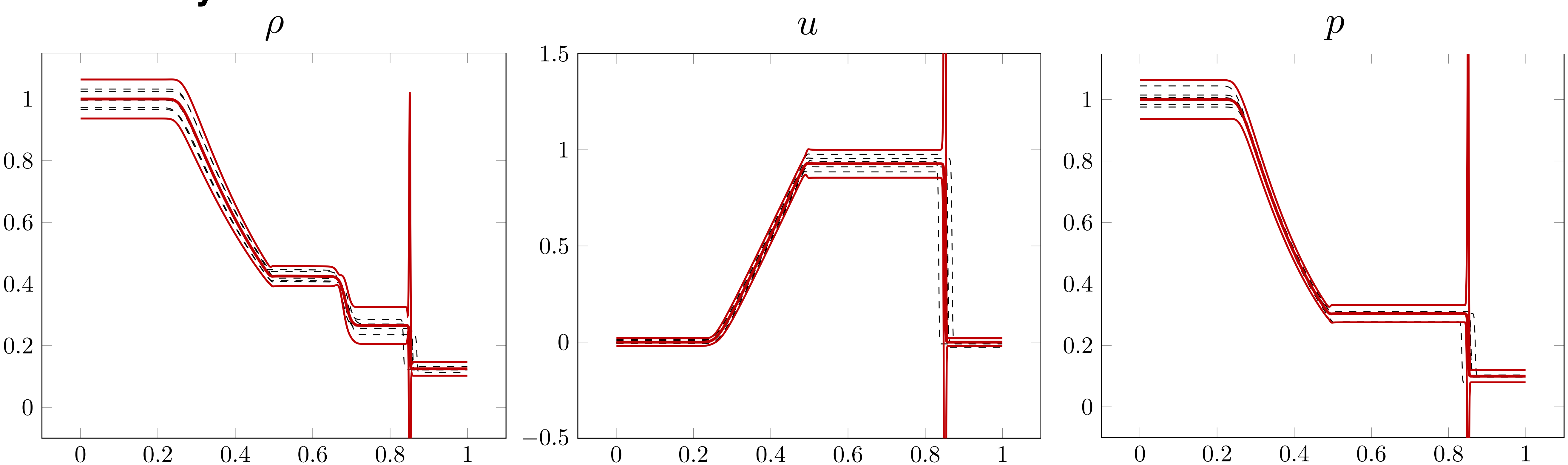
Sensitivity method without correction:



Monte Carlo method:

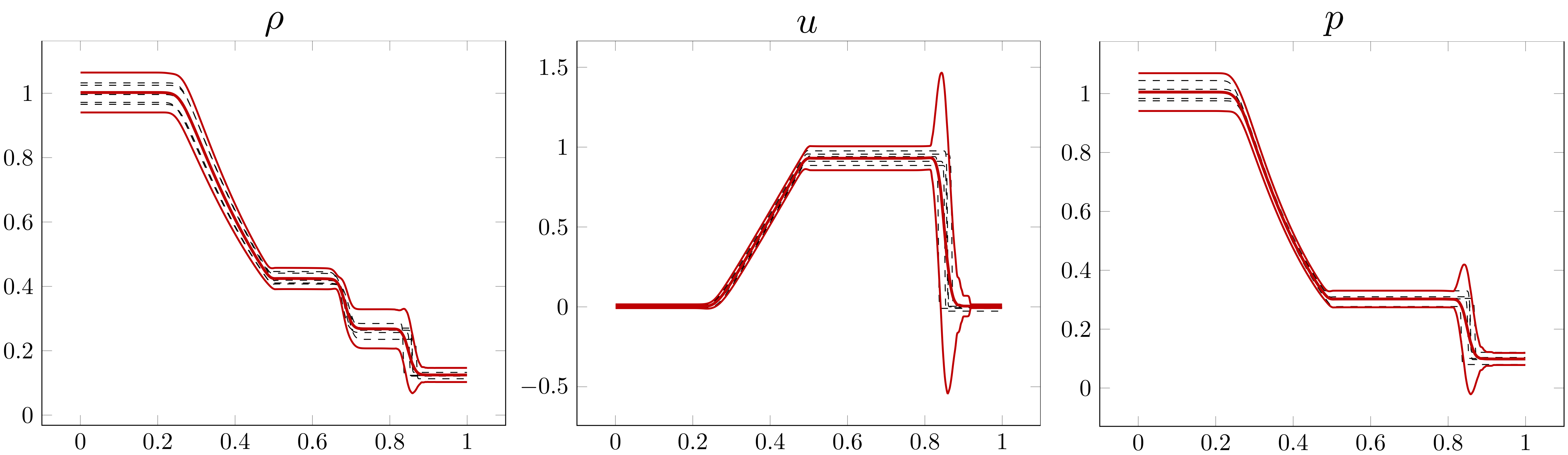


Sensitivity method without correction:

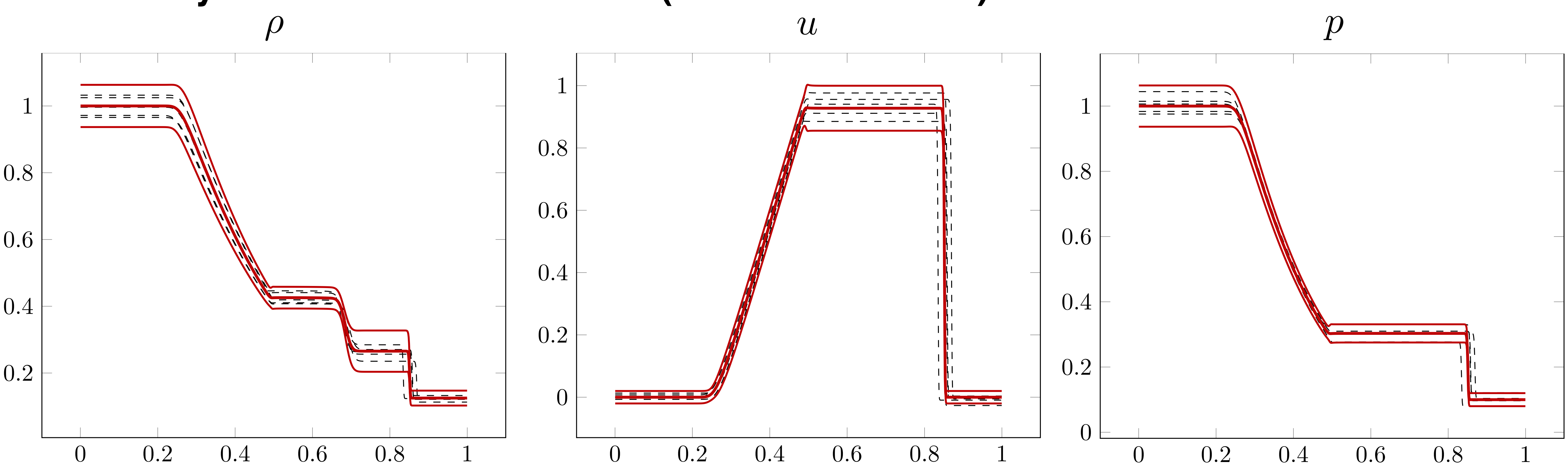


Uncertainty propagation

Monte Carlo method:

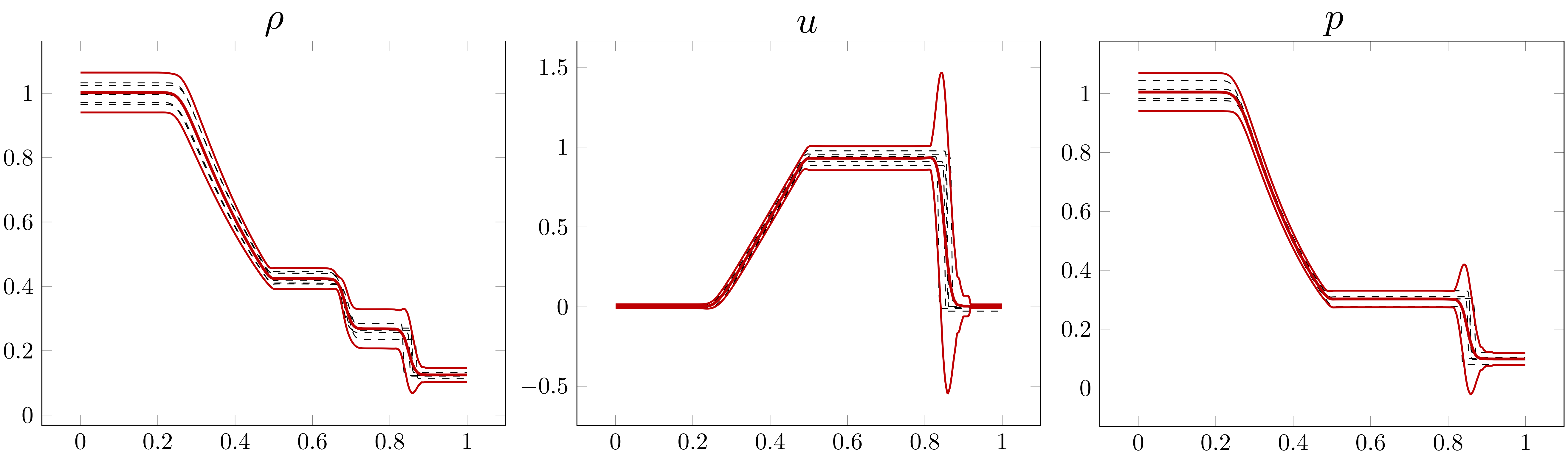


Sensitivity method with correction (diffusive method):

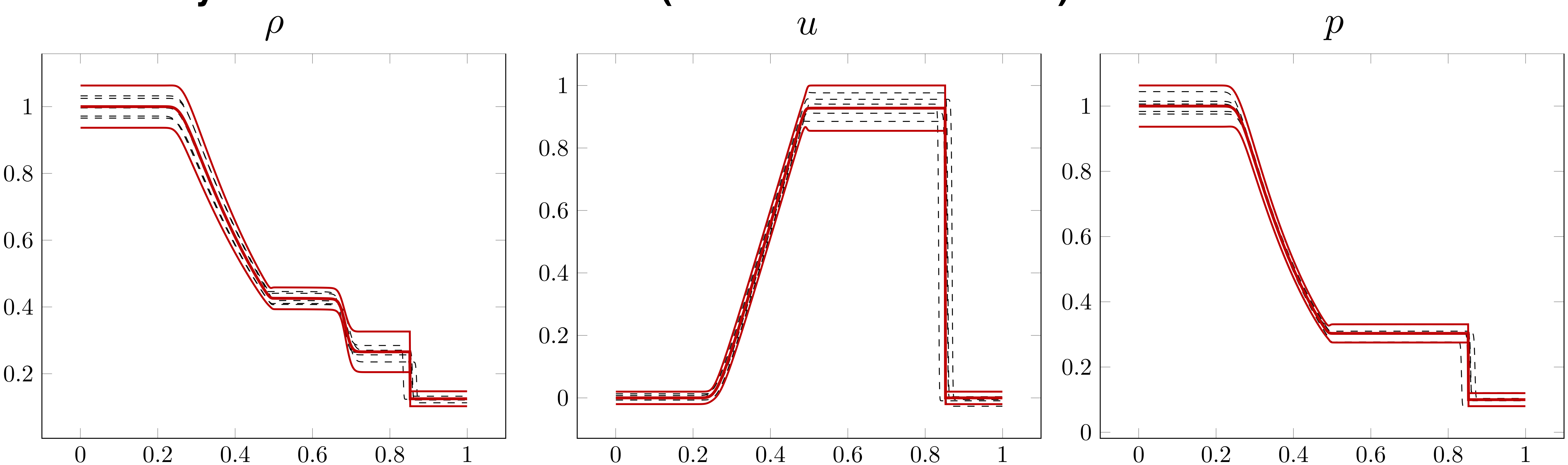


Uncertainty propagation

Monte Carlo method:



Sensitivity method with correction (anti-diffusive method):



The quasi-1D Euler equations are:

$$(1)\quad \begin{cases} \partial_t(h\rho) + \partial_x(h\rho u) = 0, \\ \partial_t(h\rho u) + \partial_x(h\rho u^2 + p) = p\partial_x h, \\ \partial_t(h\rho E) + \partial_x(hu(\rho E + p)) = 0, \end{cases}$$

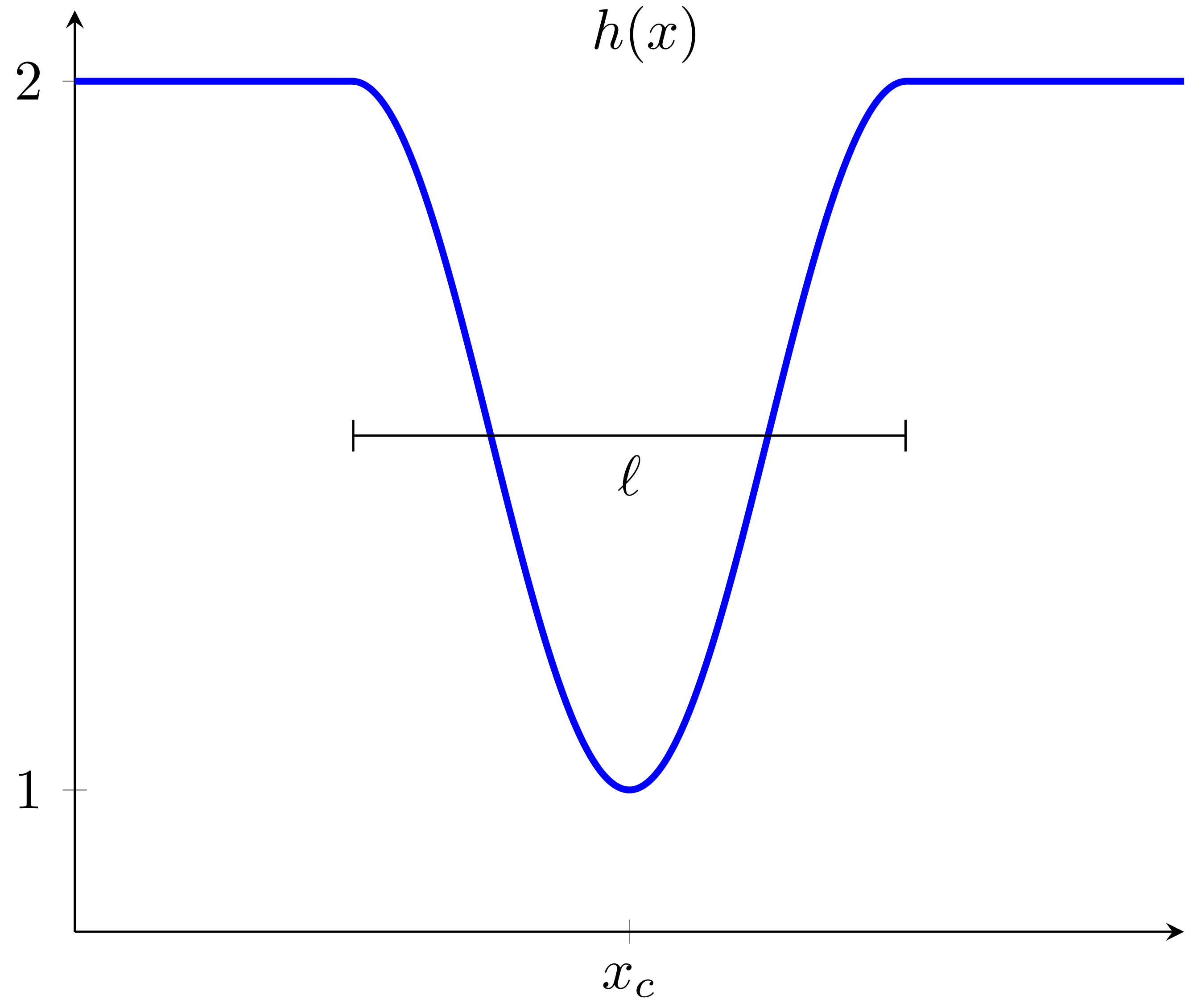
+b.c.

Cost functional: $J(\mathbf{U}) = \frac{1}{2}\|p - p^*\|^2_{L^2}$

Parameters: $\mathbf{a} = (x_c, \ell)^t$

Target pressure: $p^* = p(x_c = 0.5, \ell = 0.5)$

Gradient: $\nabla_{\mathbf{a}} J(\mathbf{U}) = \begin{bmatrix} (p - p^*, p_{x_c})_{L^2} \\ (p - p^*, p_\ell)_{L^2} \end{bmatrix}$



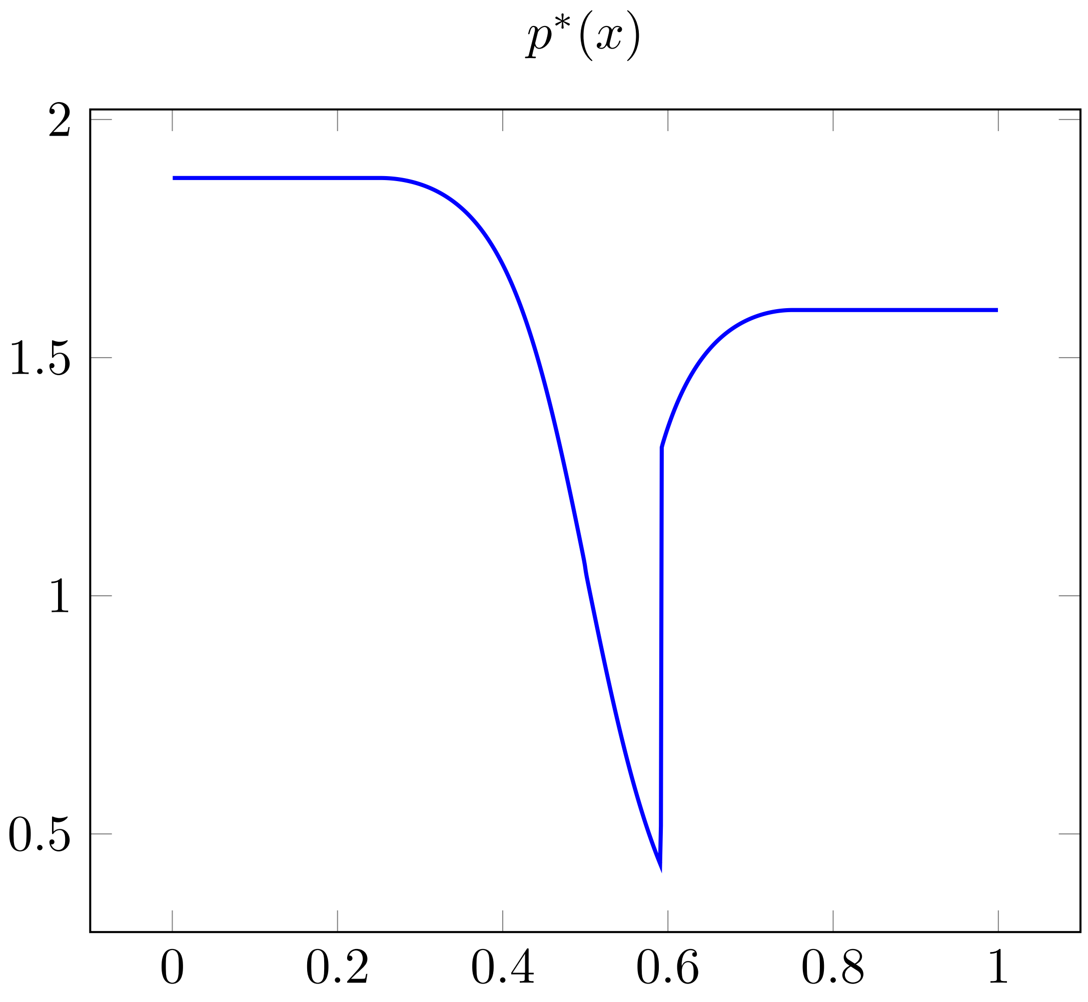
Optimization problem: $\min_{\mathbf{a} \in \mathcal{A}} J(\mathbf{U})$ subject to (1).

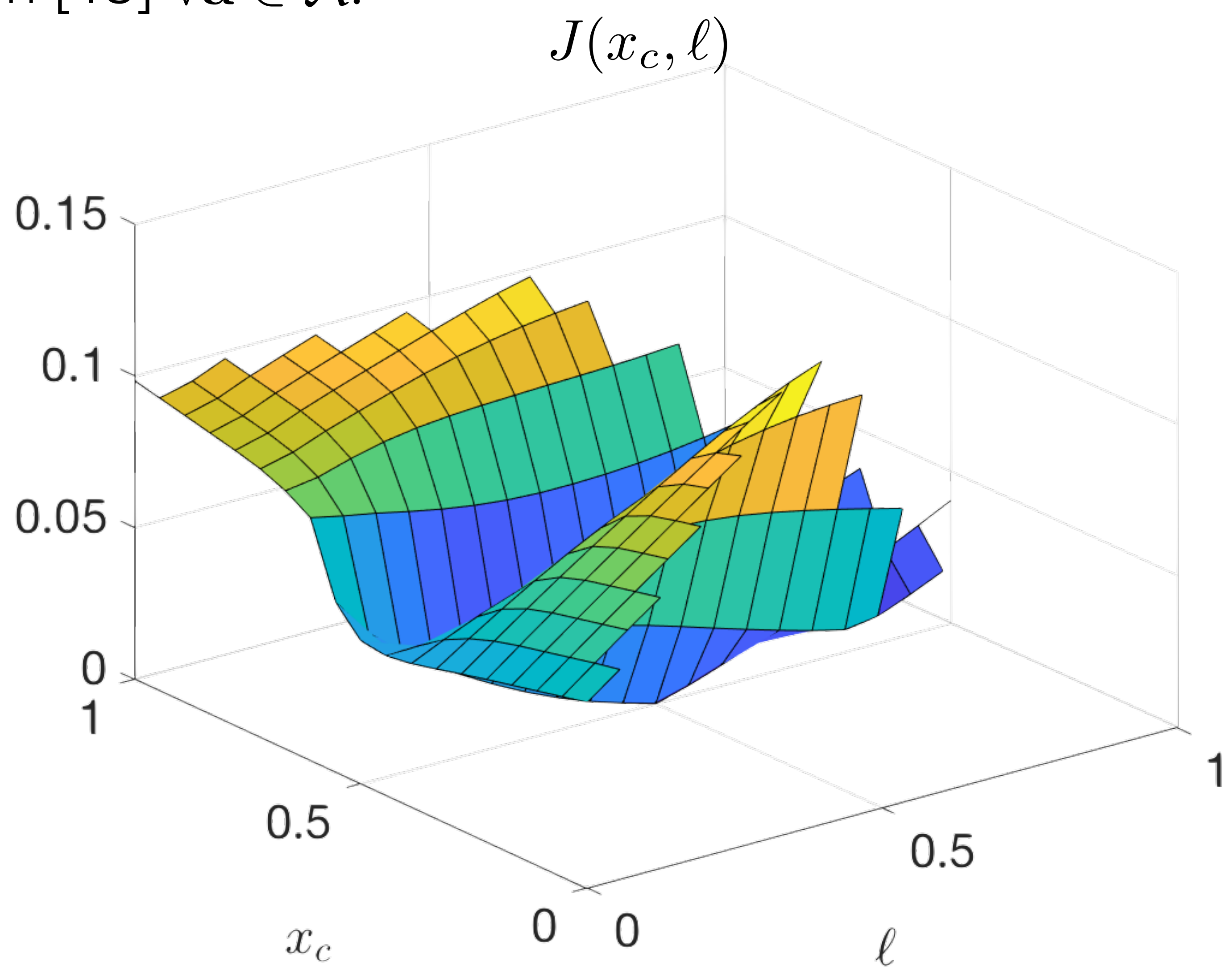
Boundary conditions:

- inlet: enthalpy H_L and total pressure $p_{tot,L}$
- outlet: pressure p_R

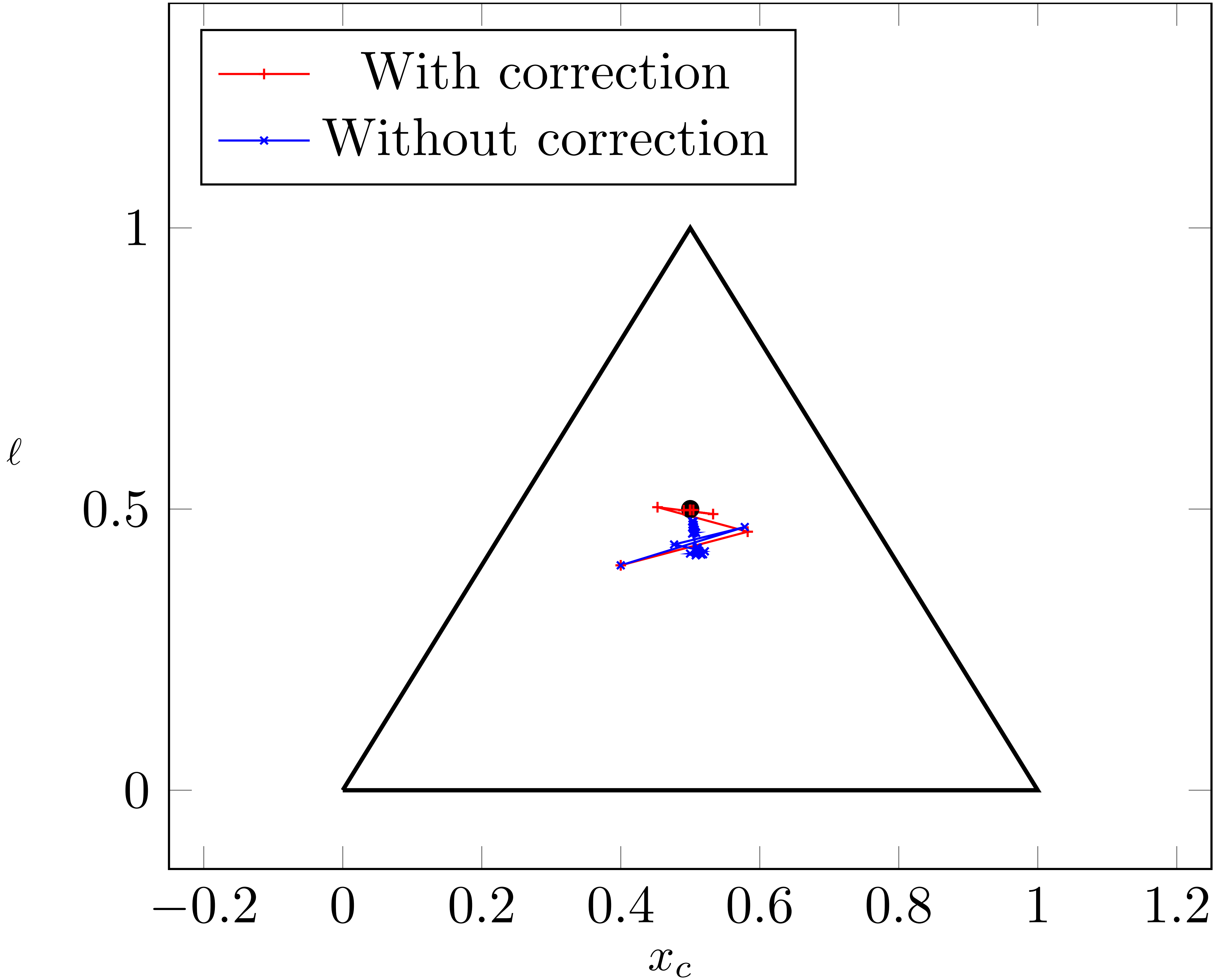
$$\begin{cases} H &= E + \dfrac{p}{\rho}, \\ p &= (\gamma - 1)\left(\rho E - \dfrac{1}{2}\rho u^2\right), \\ p_{tot} &= p + \dfrac{1}{2}\rho u^2. \end{cases}$$

These b.c. provide a discontinuous solution [13] $\forall \mathbf{a} \in \mathcal{A}$.

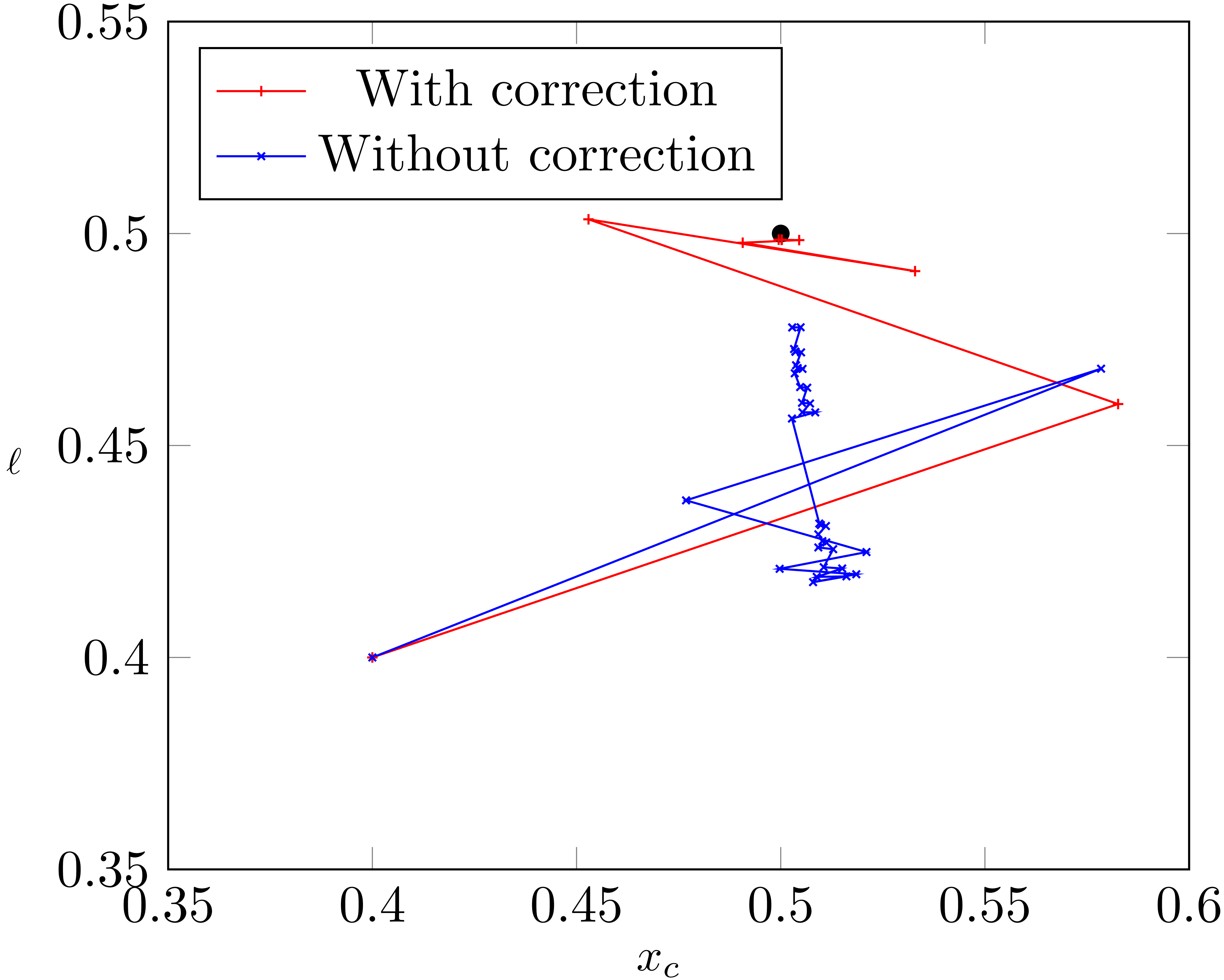




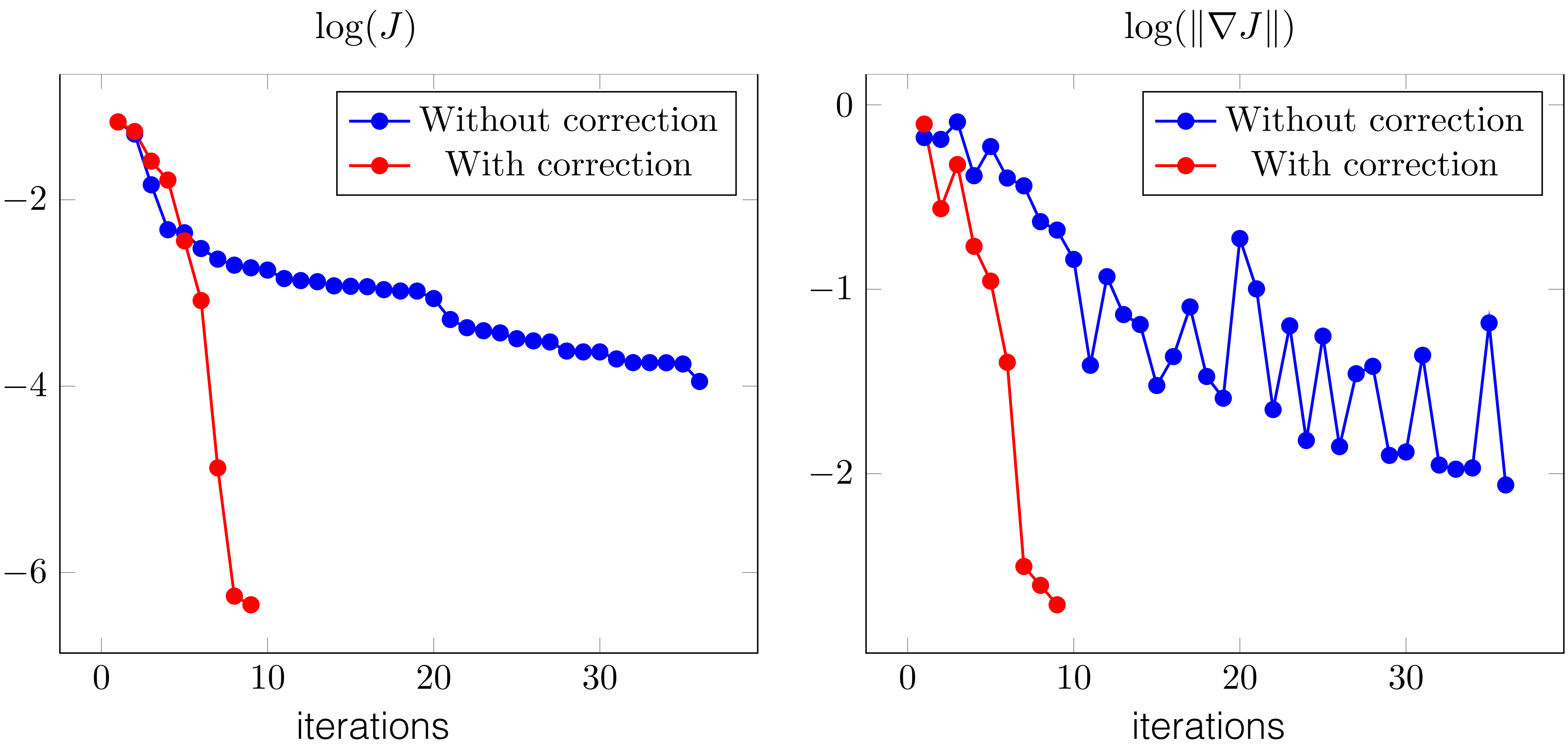
[13] Giles, M. B., & Pierce, N. A. (2001). Analytic adjoint solutions for the quasi-one-dimensional Euler equations. *Journal of Fluid Mechanics*, *426*, 327-345.


With correction
Without correction
1
0.5
0
ℓ
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
x_c

Optimization



Optimization



Conclusion and perspective

Conclusion:

- We defined a sensitivity system valid in case of discontinuous state
- The correction term is well defined
- The correction term is important in applications

Future development:

- Estimate of the variance of the shock position
- Effects of the numerical diffusion for the applications
- Extension to 2D
- Extension to different PDEs systems

Conclusion:

- We defined a sensitivity system valid in case of discontinuous state
- The correction term is well defined
- The correction term is important in applications

Future development:

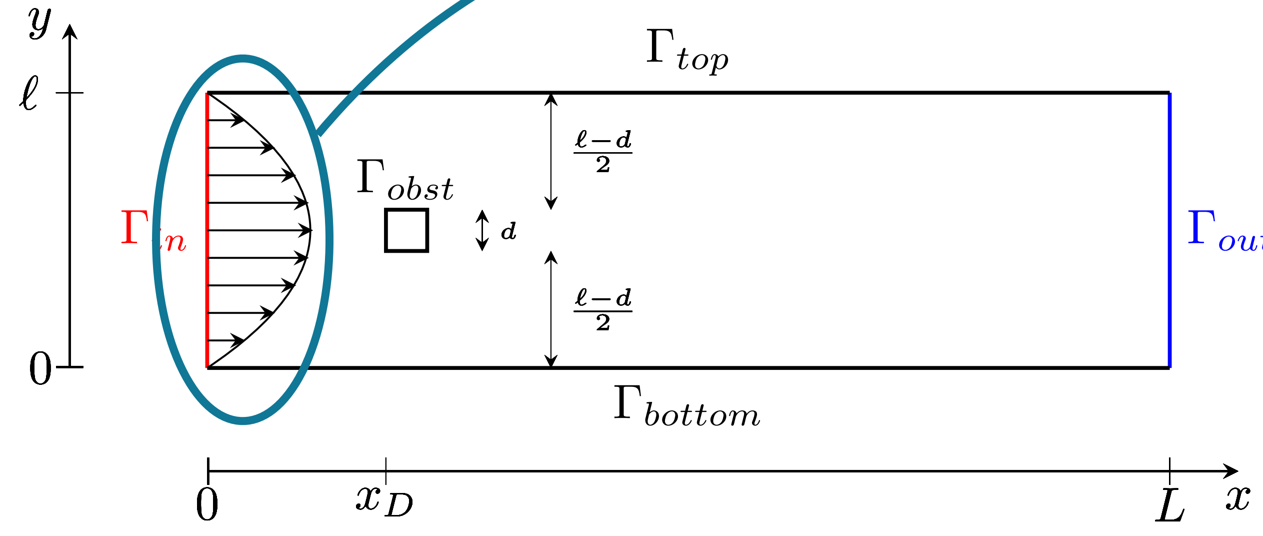
- Estimate of the variance of the shock position
- Effects of the numerical diffusion for the applications
- Extension to 2D
- **Extension to different PDEs systems**

The Navier—Stokes equations :

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \Omega, t>0, \\ \nabla\cdot\mathbf{u} = 0 & \Omega, t>0, \\ \mathbf{u}(\mathbf{x},0) = 0 & \Omega, t=0, \\ \mathbf{u} = -g(y)\mathbf{n} & \text{on } \Gamma_{in}, \\ \mathbf{u} = 0 & \text{on } \Gamma_w, \\ (\nu\nabla\mathbf{u} - pI)\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

Domain :

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$



$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$

The sensitivity equations :

$$\begin{cases} \partial_t \mathbf{u}_a - \nu \Delta \mathbf{u}_a + (\mathbf{u}_a\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_a + \nabla p_a = \overline{\mathbf{f}}_a & \Omega, t>0, \\ \nabla\cdot\mathbf{u}_a = 0 & \Omega, t>0, \\ \mathbf{u}_a(\mathbf{x},0) = 0 & \Omega, t=0, \\ \mathbf{u}_a = -g_a(y)\mathbf{n} & \text{on } \Gamma_{in}, \\ \mathbf{u}_a = 0 & \text{on } \Gamma_w, \\ (\nu\nabla\mathbf{u}_a - p_a I)\mathbf{n} = \partial_a \nu \nabla \mathbf{u}\ \mathbf{n} & \text{on } \Gamma_{out}, \end{cases}$$

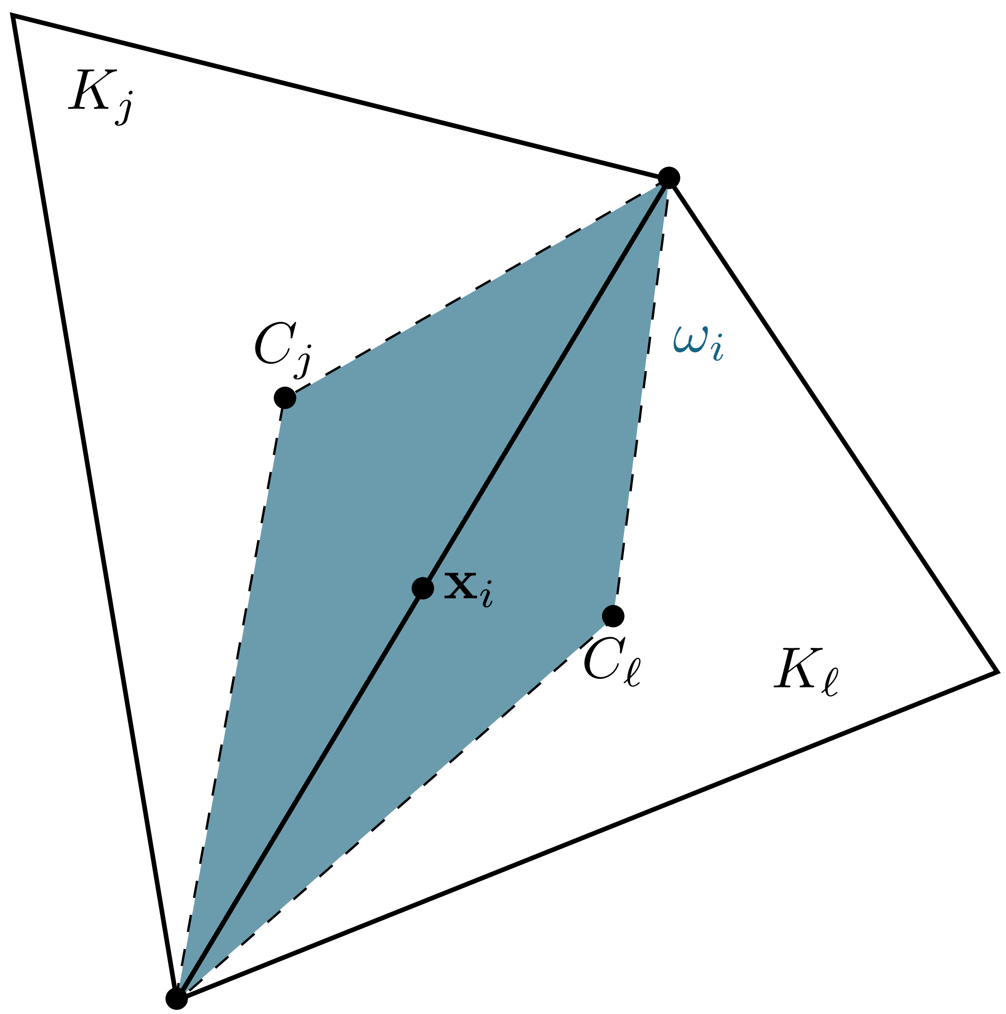
$$\overline{\mathbf{f}}_a = \partial_a \mathbf{f} + \partial_a \nu \Delta \mathbf{u}$$

Remark : these are known as the Oseen equations.

The equations are solved using the open source code **TRUST TrioCFD,** developed at CEA.

Time discretisation: **forward Euler.**

Spatial discretisation: **finite elements volumes** method (FEV).



Ingredients:

$\mathcal{T}_h$ triangulation of the domain Ω

$K_j \in \mathcal{T}_h$ triangles $j = 1, \dots, N_T$

$\mathbf{x}_i$ nodes $i = 1, \dots, N_N$

ω_i control volume

Spaces:

$$Q_h = \{q_h : \forall K \in \mathcal{T}_h,\ q_h|_K \in P_0(K)\},$$

$$V_h = \{w_h \text{ continuous in } \mathbf{x}_i : \forall K \in \mathcal{T}_h,\ w_h|_K \in P_1(K)\},$$

$$\mathbf{V}_h = \{\mathbf{w}_h = (w_x, w_y)^t : w_x,\ w_y \in V_h\}.$$

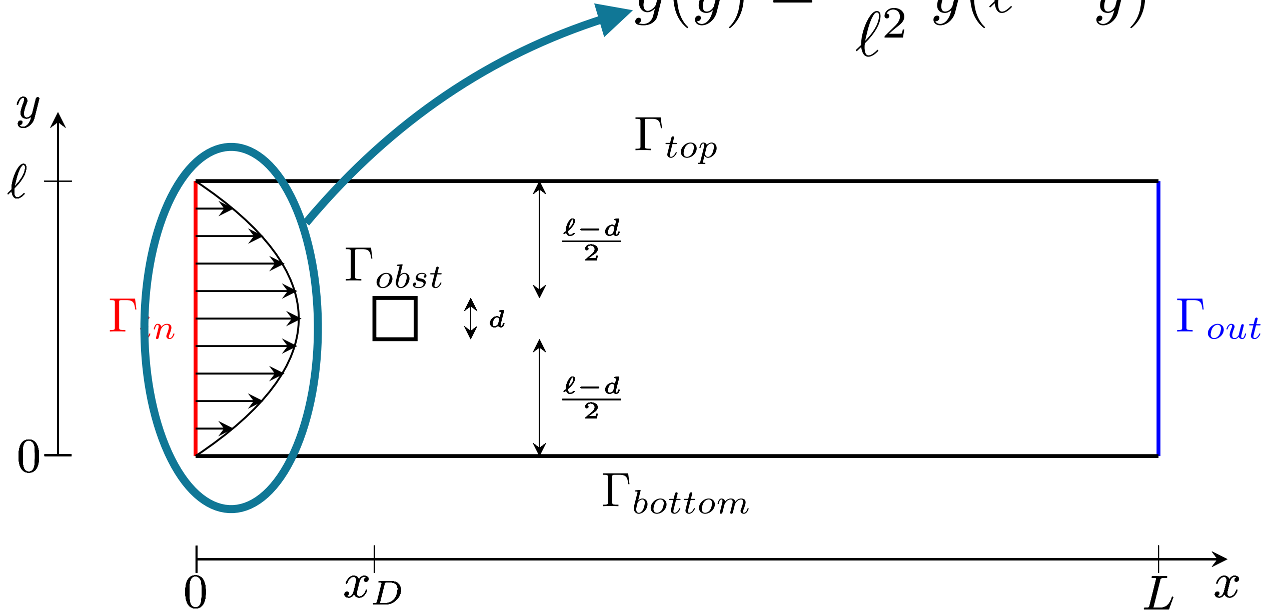
Basis functions : $\varphi_i(\mathbf{x}_j) = \delta_{i,j}$ for V_h $\quad \chi_K$ for Q_h

Remark : $V_h \not\subset H^1(\Omega)$

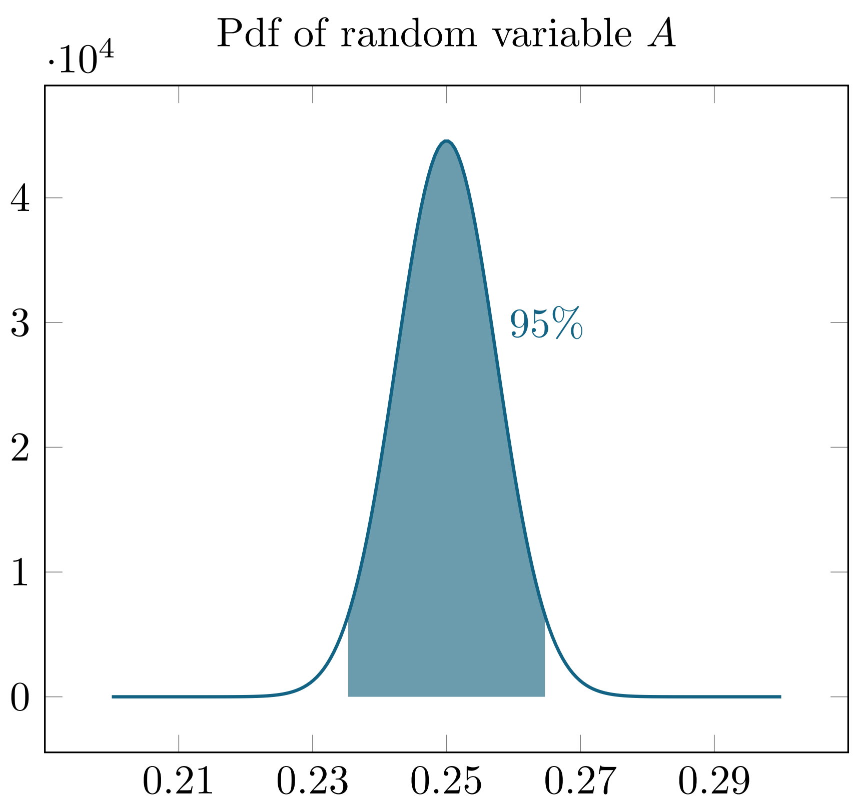
Domain :

uncertain parameter

$$g(y) = \frac{4A}{\ell^2} y(\ell - y)$$



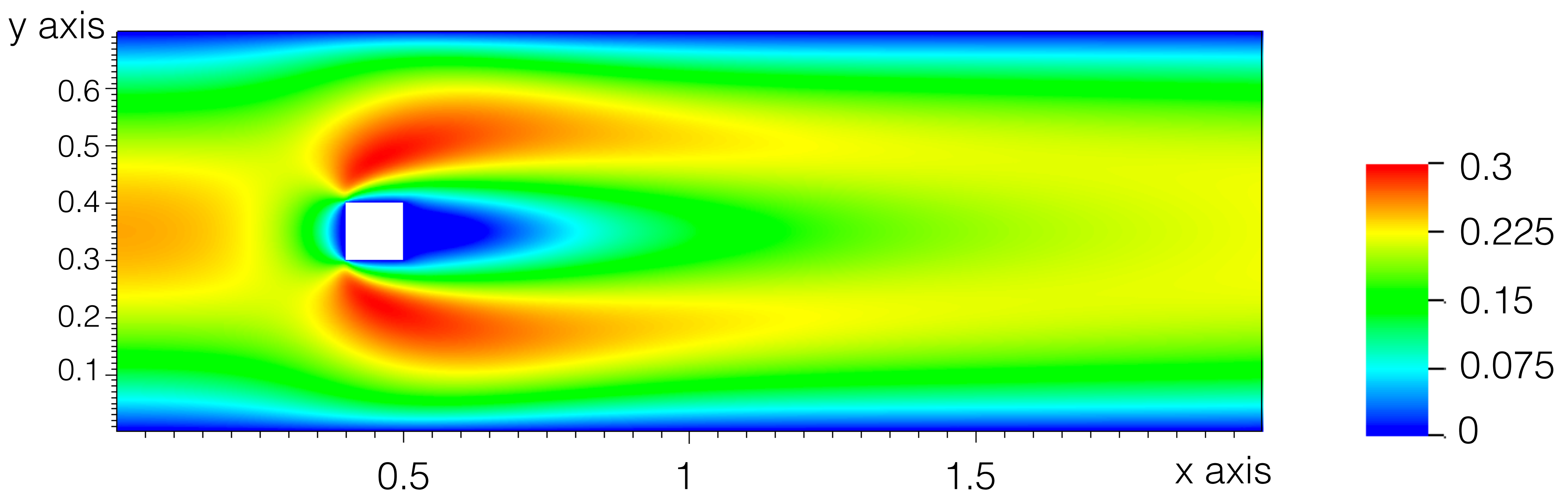
$$\Gamma_w = \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$$



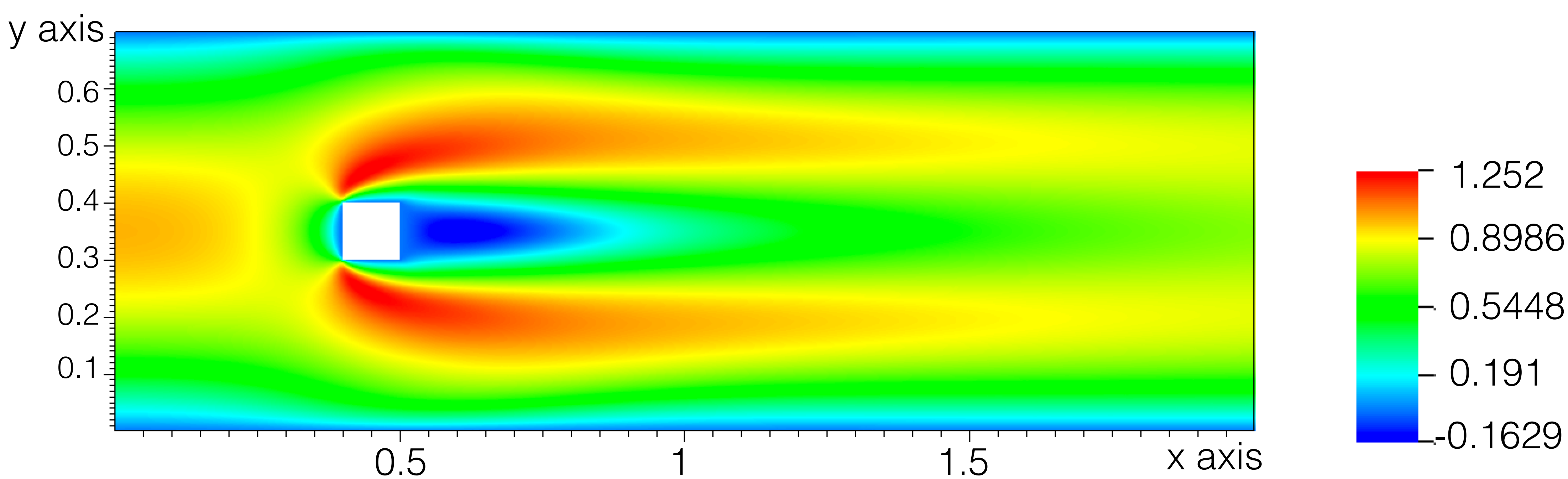
Numerical results

Steady case : x-component of the velocity and its sensitivity

State

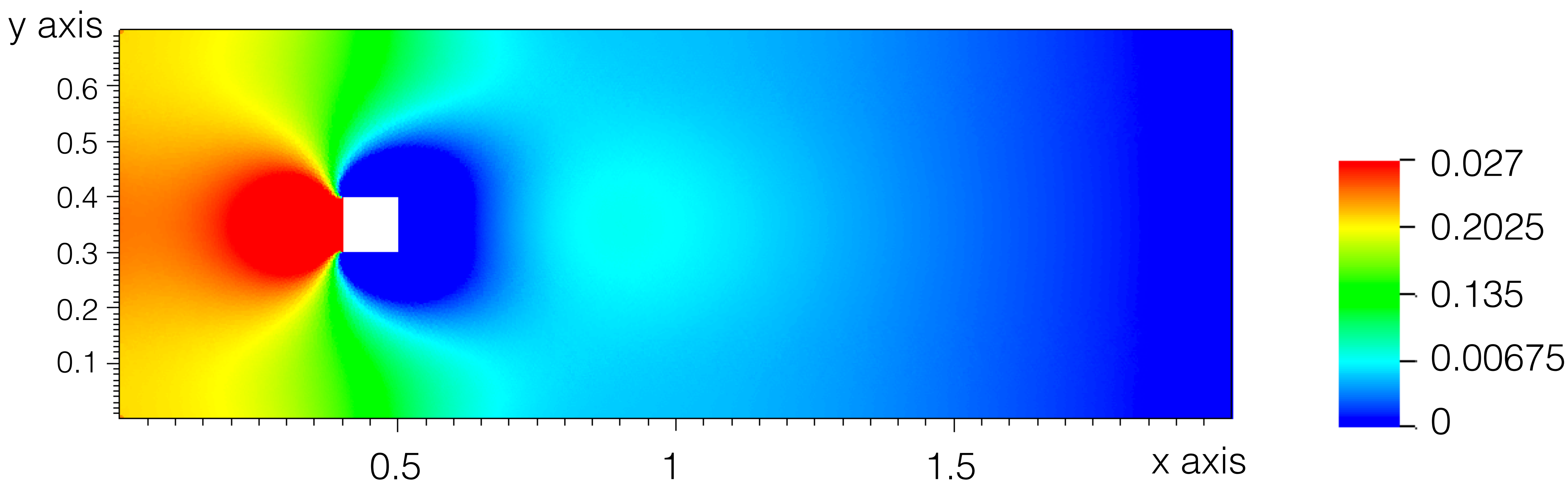


Sensitivity

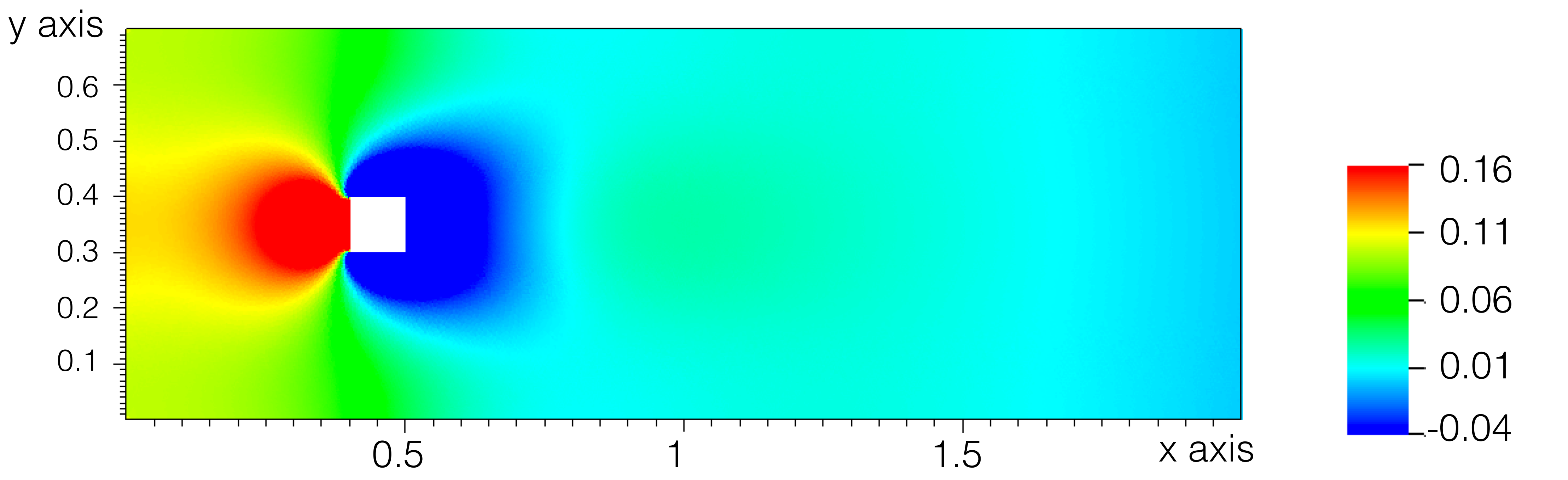


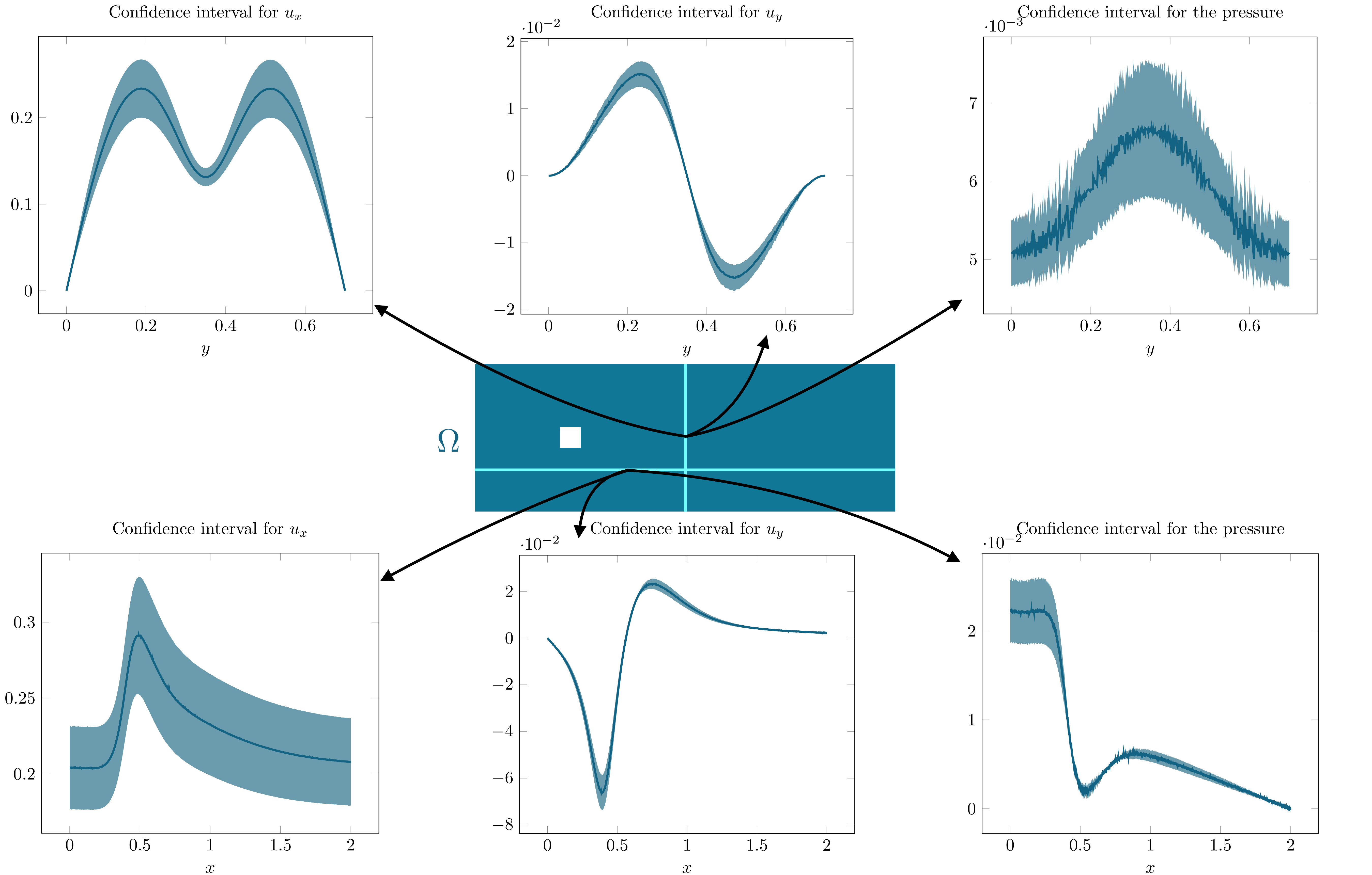
Steady case : pressure and its sensitivity

State



Sensitivity




Confidence interval for u_x
0.2
0.1
0
0
0.2
0.4
0.6
y
Confidence interval for u_y
$\cdot 10^{-2}$
2
1
0
−1
−2
0
0.2
0.4
0.6
y
Confidence interval for the pressure
$\cdot 10^{-3}$
7
6
5
0
0.2
0.4
0.6
y
Ω
Confidence interval for u_x
0.3
0.25
0.2
0
0.5
1
1.5
2
x
Confidence interval for u_y
$\cdot 10^{-2}$
2
0
−2
−4
−6
−8
0
0.5
1
1.5
2
x
Confidence interval for the pressure
$\cdot 10^{-2}$
2
1
0
0
0.5
1
1.5
2
x

Thank you

for your attention!