Sensitivity analysis for nonlinear hyperbolic systems of conservation laws

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Outline of the talk

- Sensitivity analysis
- Sensitivity analysis for hyperbolic equations
- Riemann problem for the Euler equations and their sensitivity
- Classical numerical schemes
- Anti-diffusive numerical schemes
- Applications

- **<u>Sensitivity analysis</u>**
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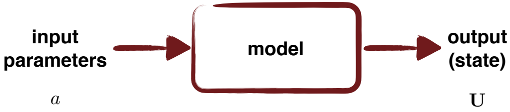
Sensitivity Analysis

Sensitivity analysis: study of how changes in the **inputs** of a model affect the **outputs**



input parameters a → model → output (state) $\mathbf{U}$

Sensitivity: $$\frac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$$

Two approaches

Discretise then differentiate

- analytical state model
- ↓ discretise
- discrete state model
- ↓ differentiate
- discrete sensitivity model

Differentiate then discretise

- analytical state model
- ↓ differentiate
- analytical sensitivity model
- ↓ discretise
- discrete sensitivity model

analytical sensitivity model

no discretisation of computational facilitators

could lead to inconsistent gradients

Standard techniques of sensitivity analysis

Standard techniques of sensitivity analysis call for the differentiation of the state system:

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 & \Omega \times (0, T), \\ \mathbf{U}(x, 0) = \mathbf{g}(x) & \Omega, \end{cases}$$

[1] Bardos, C., Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

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For the **Burgers' equation**:

$$\mathbf{F}(\mathbf{U}) = \frac{u^2}{2} \qquad \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = u u_a$$

This can be done under **hypotheses of regularity** of the state ***U*** [1].

If these techniques are applied to hyperbolic equations, **Dirac delta functions** will appear in the sensitivity.

[1] Bardos, C., Pironneau, O. (2002). A formalism for the differentiation of conservation laws. *Comptes Rendus Mathematique*, *335*(10), 839-845.

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Definition of the source term

In order to have a sensitivity system which is valid also when the state is discontinuous, we add a correction term [2]:

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S} & \Omega \times (0, T), \\ \mathbf{U}_a(x, 0) = \mathbf{g}_a(x) & \Omega, \end{cases}$$

defined as follows:

$$\mathbf{S} = \sum_{k=1}^{N_s} \boldsymbol{\rho}_k \delta(x - x_{k,s}(t)),$$

N_s: number of discontinuities

$x_{k,s}$: position of the k-th discontinuity

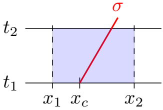
$\boldsymbol{\rho}_k$: amplitude of the k-th correction (to be computed)

Remark: a **shock detector** is necessary to discretise such source term.

[2] Guinot, V., Delenne, C., Cappelaere, B. (2009). An approximate Riemann solver for sensitivity equations with discontinuous solutions. *Advances in Water Resources*, *32*(1), 61-77.

Definition of the source term

To compute the amplitude of the correction, we consider an infinitesimal control volume containing a single discontinuity:



By integrating the sensitivity equations with the source term on the control volume, one has:

$$\boldsymbol{\rho}_k = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + \mathbf{F}_a^+ - \mathbf{F}_a^-$$

Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma_k = \mathbf{F}^+ - \mathbf{F}^-$

Differentiating them w.r.t. the parameter:

$$(\mathbf{U}_a^+ - \mathbf{U}_a^-)\sigma_k + (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x\mathbf{U}^+ - \partial_x\mathbf{U}^-)\partial_a x_{k,s}(t) =$$

$$= \mathbf{F}_a^+ - \mathbf{F}_a^- + \left(\frac{\partial\mathbf{F}(\mathbf{U}^+)}{\partial\mathbf{U}}\partial_x\mathbf{U}^+ - \frac{\partial\mathbf{F}(\mathbf{U}^-)}{\partial\mathbf{U}}\partial_x\mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

Finally, we obtain the following amplitude:

$$\boldsymbol{\rho}_k = (\mathbf{U}^+ - \mathbf{U}^-)\sigma_{k,a} + \sigma_k(\partial_x\mathbf{U}^+ - \partial_x\mathbf{U}^-)\partial_a x_{k,s}(t) - \left(\frac{\partial\mathbf{F}(\mathbf{U}^+)}{\partial\mathbf{U}}\partial_x\mathbf{U}^+ - \frac{\partial\mathbf{F}(\mathbf{U}^-)}{\partial\mathbf{U}}\partial_x\mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

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The Euler equations are:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(u(\rho E + p)) = 0, \end{cases}$$

Eigenvalues:	Eigenvectors:
$\lambda_1(\mathbf{U}) = u - c,$	$\mathbf{r}_1(\mathbf{U}) = (1, u - c, H - uc)^t,$
$\lambda_2(\mathbf{U}) = u,$	$\mathbf{r}_2(\mathbf{U}) = (1, u, \frac{u^2}{2})^t,$
$\lambda_3(\mathbf{U}) = u + c.$	$\mathbf{r}_3(\mathbf{U}) = (1, u + c, H + uc)^t.$

Genuinely nonlinear

The Riemann problem for Euler equations

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Linearly degenerate

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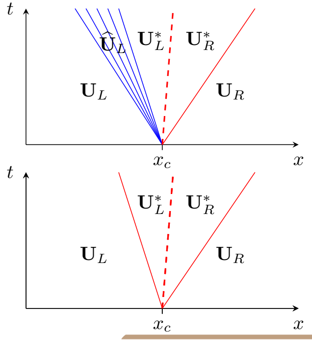

The Riemann problem for the sensitivity equations

The sensitivity system is:

$$\begin{cases} \partial_t \rho_a + \partial_x (\rho u)_a = S_1, \\ \partial_t (\rho u)_a + \partial_x (\rho_a u^2 + 2\rho u u_a + p_a) = S_2, \\ \partial_t (\rho E)_a + \partial_x (u_a(\rho E + p) + u((\rho E)_a + p_a)) = S_3, \end{cases}$$

Eigenvalues:

$$\lambda_1(\mathbf{U}) = u - c,$$
$$\lambda_2(\mathbf{U}) = u,$$
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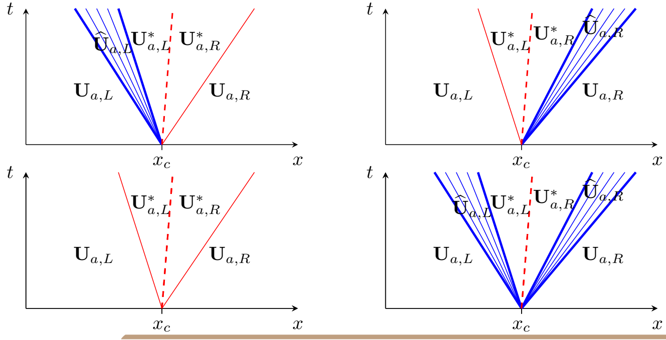

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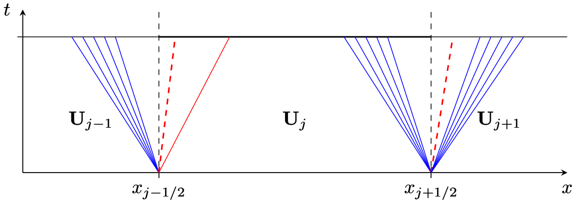
Godunov-type schemes

approximate Riemann solvers are used

Step 0 : initial data discretisation

Step 1 : solution of a Riemann problem for each interface $x_{j-1/2}$ obtaining $\mathbf{v}(x, t^{n+1})$

Step 2 : average $\mathbf{V}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(x, t^{n+1}) \mathrm{d}x$



Classical numerical schemes

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U}, \mathbf{U}_a) = \mathbf{S}(\mathbf{U}) \end{cases} \qquad \mathbf{S}(\mathbf{U}) = \sum_{k=1}^{N_s} \sigma_{a,k}(\mathbf{U}_k^+ - \mathbf{U}_k^-)$$

Remark: HLL-type schemes cannot be used for the state, two intermediate star states are necessary to have a well-defined the source term for the sensitivity.

Approximate Riemann solver for the state

- First order Roe-type scheme

$$\lambda_1^{ROE} = \tilde{u} - \tilde{c} \qquad \lambda_2^{ROE} = \tilde{u} \qquad \lambda_3^{ROE} = \tilde{u} + \tilde{c}$$ Roe-averaged eigenvalues

$$\mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{3} \alpha_i \tilde{\mathbf{r}}_i$$ decomposition along Roe-averaged eigenvectors

$$\mathbf{U}_L^* = \mathbf{U}_L + \alpha_1 \tilde{\mathbf{r}}_1 \qquad \mathbf{U}_R^* = \mathbf{U}_R - \alpha_3 \tilde{\mathbf{r}}_3$$

[3] Bouchut, F. (2004). Nonlinear stability of finite Volume Methods for hyperbolic conservation laws: And Well-Balanced schemes for sources. Springer Science & Business Media.

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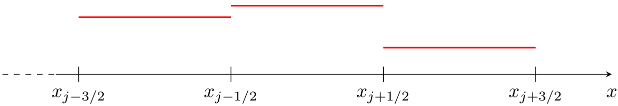
Approximate Riemann solver for the state

- First order Roe-type scheme
- Second order Roe-type scheme

  Time discretisation: two-step Runge-Kutta method

  Space discretisation: MUSCL-type scheme [3]



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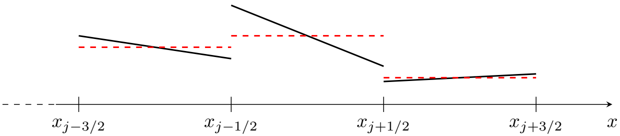

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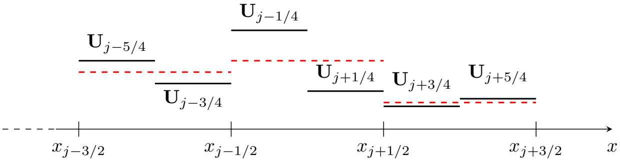

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Approximate Riemann solvers for the sensitivity

- HLL-type scheme: simpler structure that the state solver.

HLL consistency conditions yield:

$$\mathbf{U}^*_{a,j-1/2} = \frac{1}{\lambda_3^{ROE} - \lambda_1^{ROE}} \left(\lambda_3^{ROE} \mathbf{U}^n_{a,j} - \lambda_1^{ROE} \mathbf{U}^n_{a,j-1} - \mathbf{F}_a(\mathbf{U}_j, \mathbf{U}_{a,j}) + \mathbf{F}_a(\mathbf{U}_{j-1}, \mathbf{U}_{a,j-1}) + \mathbf{S}_{j-1/2} \right)$$

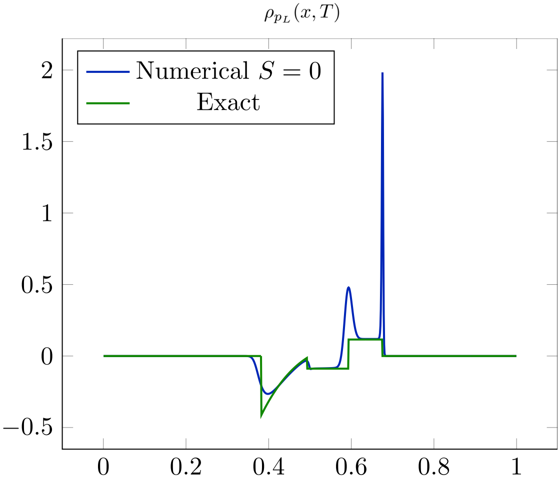
$$\begin{aligned} \mathbf{S}_{j-1/2} = & \partial_a \lambda^{ROE}_{1,j-1/2} (\mathbf{U}^*_{L,j-1/2} - \mathbf{U}_{j-1}) d_{1,j-1/2} \\ & + \partial_a \lambda^{ROE}_{2,j-1/2} (\mathbf{U}^*_{R,j-1/2} - \mathbf{U}^*_{L,j-1/2}) \\ & + \partial_a \lambda^{ROE}_{3,j-1/2} (\mathbf{U}_j - \mathbf{U}^*_{R,j-1/2}) d_{3,j-1/2} \end{aligned}$$

- HLLC-type scheme: same structure as the state.

HLL consistency conditions + Rankine-Hugoniot conditions. Equivalent to:

$$\mathbf{U}^*_{a,L} = \mathbf{U}_{a,L} + \alpha_{1,a} \tilde{\mathbf{r}}_1 + \alpha_1 \tilde{\mathbf{r}}_{1,a} \qquad \mathbf{U}^*_{a,R} = \mathbf{U}_{a,R} - \alpha_{3,a} \tilde{\mathbf{r}}_3 - \alpha_3 \tilde{\mathbf{r}}_{3,a}$$

Classical numerical schemes



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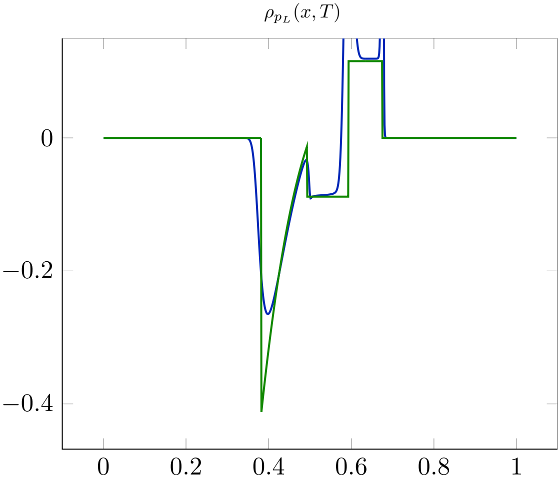

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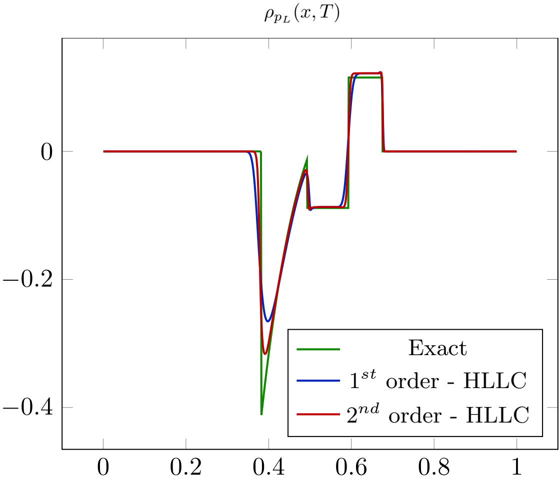
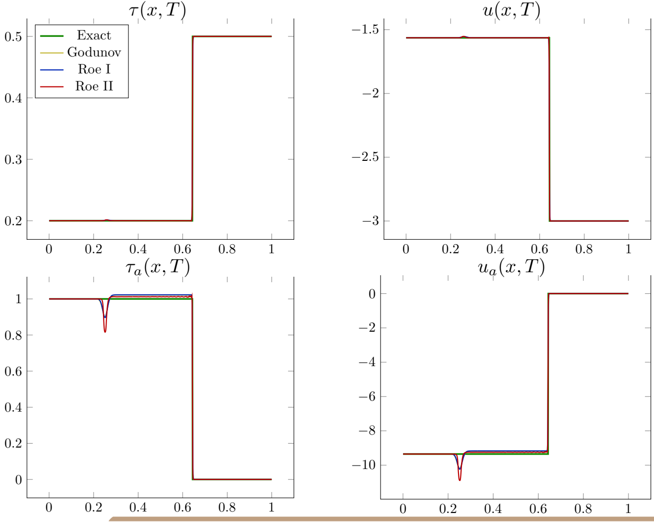


Isolated shock for the *p*-system



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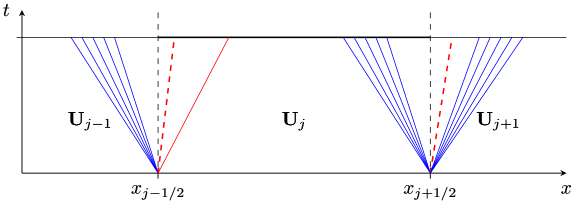
Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : ~~average~~

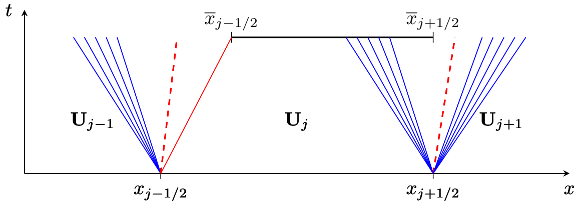


Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed [4]



$$\overline{x}_{j-1/2} = x_{j-1/2} + \sigma_{j-1/2}\Delta t$$

[4] Chalons, C., Goatin, P. (2008). Godunov scheme and sampling technique for computing phase transitions in traffic flow modeling. *Interfaces and Free Boundaries*, *10*(2), 197-221.

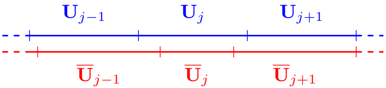
Scheme without numerical diffusion

Step 0 : initial data discretisation

Step 1 : solution of the Riemann problems, one for each interface

Step 2 : definition of a staggered mesh on which the average is performed

Step 3 : projection on the initial mesh [5]

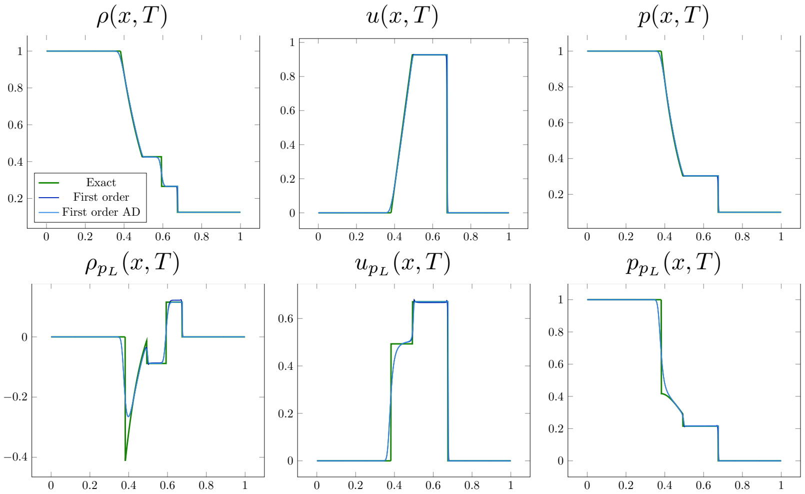


$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } \alpha \in \left(0, \frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j} & \text{if } \alpha \in \left[\frac{\Delta t}{\Delta x} \max(\sigma_{j-1/2}, 0), 1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } \alpha \in \left[1 + \frac{\Delta t}{\Delta x} \min(\sigma_{j+1/2}, 0), 1\right). \end{cases}$$

$$\alpha \sim \mathcal{U}([0,1])$$

[5] Glimm, J. (1965). Solutions in the large for nonlinear hyperbolic systems of equations. *Communications on pure and applied mathematics*, *18*(4), 697-715.

Numerical results



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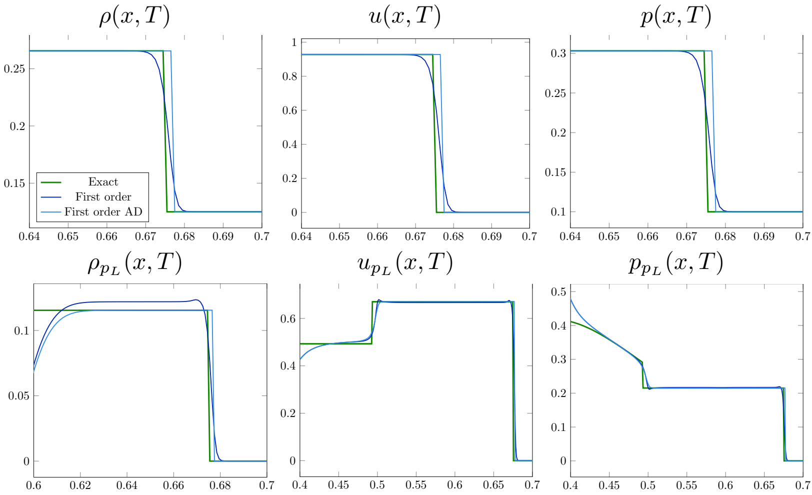

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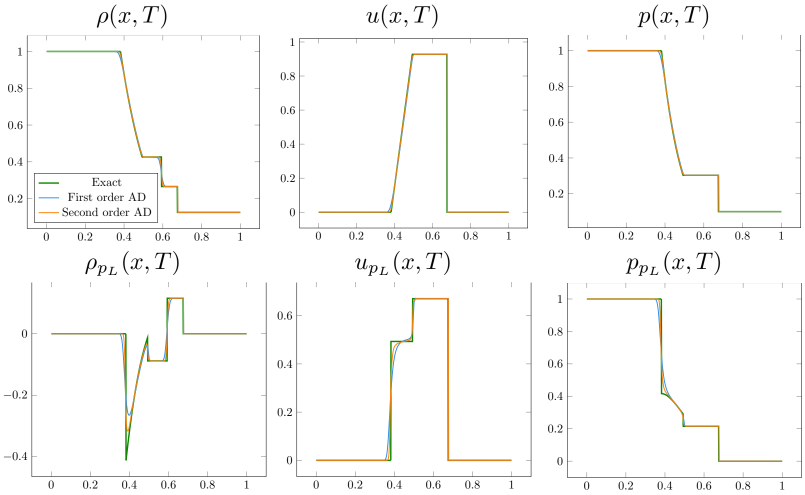
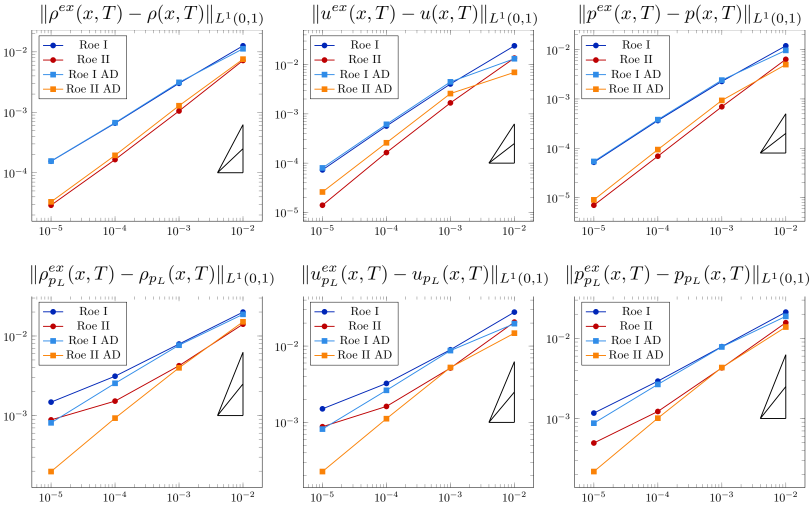


Convergence



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Uncertainty Quantification

Let **a** be a random vector, with the following average and variance:

$$\mu_{\mathbf{a}} = \begin{bmatrix} \mu_{a_1} \\ \vdots \\ \mu_{a_M} \end{bmatrix}, \qquad \sigma_{\mathbf{a}} = \begin{bmatrix} \sigma^2_{a_1} & \mathrm{cov}(a_1, a_2) & \dots & \mathrm{cov}(a_1, a_M) \\ \mathrm{cov}(a_1, a_2) & \sigma^2_{a_2} & \dots & \mathrm{cov}(a_2, a_M) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(a_1, a_M) & \dots & & \sigma^2_{a_M} \end{bmatrix},$$

Aim: determine a **confidence interval** $CI_X = [\mu_X - \kappa\sigma_X, \mu_X + \kappa\sigma_X]$

Monte Carlo approach: *N* samples of the stateX_k

$$\mu_X = \frac{1}{N}\sum_{k=1}^{N} X_k \qquad \sigma^2_X = \frac{1}{N-1}\sum_{k=1}^{N} |\mu_X - X_k|^2$$

Uncertainty Quantification

Sensitivity approach: let us consider the following first order Taylor expansion

$$X(\mathbf{a}) = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} (a_i - \mu_{a_i}) X_{a_i}(\mu_{\mathbf{a}}) + o(\|\mathbf{a}\|^2).$$

Then, computing, the average one has:

$$\mu_X = E[X(\mathbf{a})] = X(\mu_{\mathbf{a}}) + \sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}}) E[a_i - \mu_{a_i}] = X(\mu_{\mathbf{a}}),$$

And for the variance:

$$\sigma_X^2 = E[(X(\mathbf{a}) - \mu_X)^2] = E\left[\left(\sum_{i=1}^{M} X_{a_i}(\mu_{\mathbf{a}})(a_i - \mu_{a_i})\right)^2\right] =$$

$$= \sum_{i=1}^{M} X_{a_i}^2(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})^2] + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i}(\mu_{\mathbf{a}}) X_{a_j}(\mu_{\mathbf{a}}) E[(a_i - \mu_{a_i})(a_j - \mu_{a_j})].$$

Therefore, we have the following first order estimates:

$$\mu_X = X(\mu_{\mathbf{a}}), \quad \sigma_X^2 = \sum_{i=1}^{M} X_{a_i}^2 \sigma_{a_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} X_{a_i} X_{a_j} \mathrm{cov}(a_i, a_j).$$

Test case:

Riemann problem with uncertain parameters: $\mathbf{a} = (\rho_L, \rho_R, u_L, u_R, p_L, p_R)^t$

with the following average and covariance matrix:

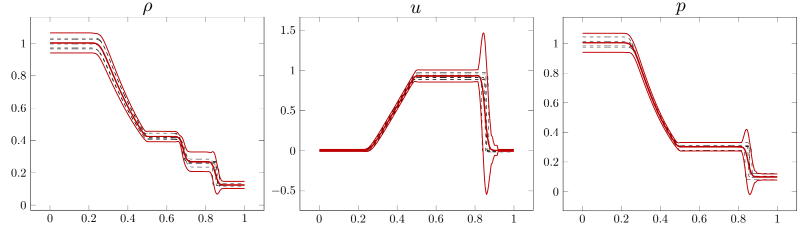
$$\mu_{\mathbf{a}} = (1, 0.125, 0, 0, 1, 0.1)^t, \quad \sigma_{\mathbf{a}} = \mathrm{diag}(0.001, 0.000125, 0.0001, 0.0001, 0.001, 0.0001).$$

Since the covariance matrix is diagonal, the previous estimate is simplified:

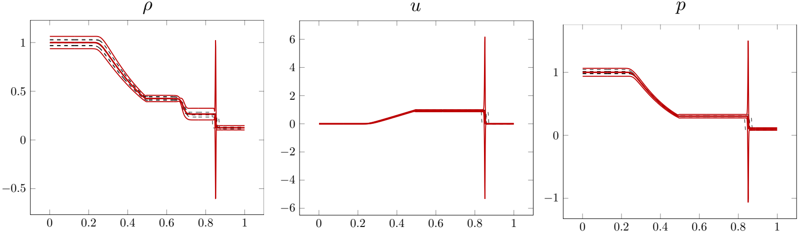
$$\sigma_X^2 = \sum_{i=1}^{6} X_{a_i}^2 \sigma_{a_i}^2$$

Uncertainty Quantification

Monte Carlo method:

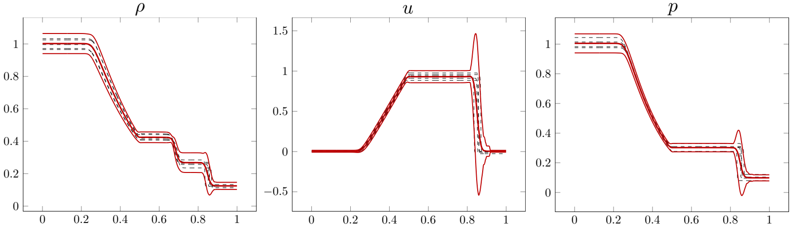


Sensitivity method without correction:

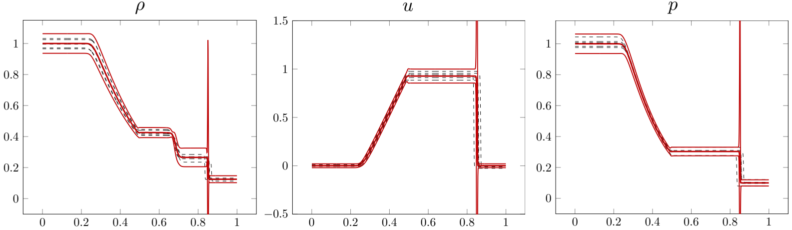


Uncertainty Quantification

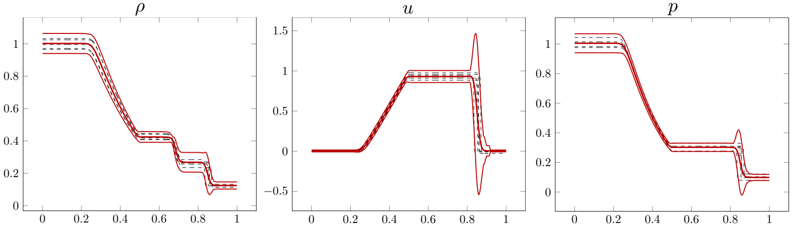
Monte Carlo method:



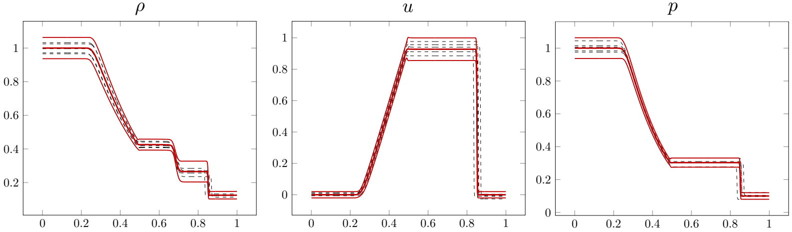
Sensitivity method without correction:



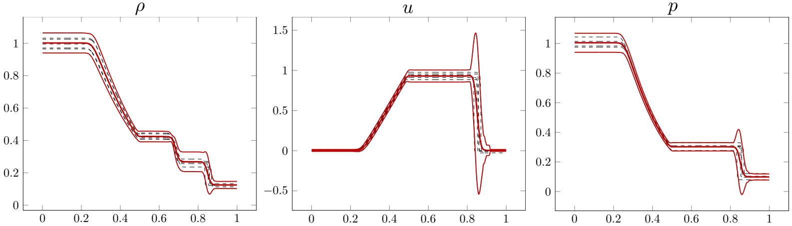
Monte Carlo method:



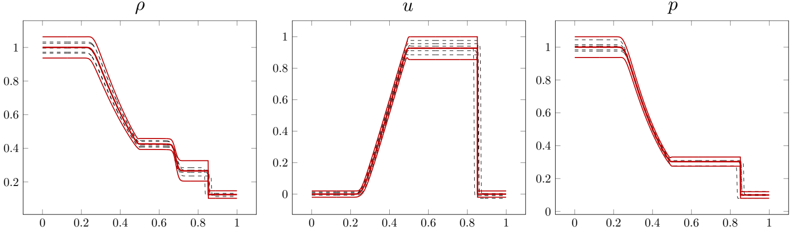
Sensitivity method with correction (diffusive method):



Monte Carlo method:



Sensitivity method with correction (anti-diffusive method):



Optimisation

The quasi-1D Euler equations are:

$$(1)\quad \begin{cases} \partial_t(h\rho) + \partial_x(h\rho u) = 0, \\ \partial_t(h\rho u) + \partial_x(h\rho u^2 + p) = p\partial_x h, \\ \partial_t(h\rho E) + \partial_x(hu(\rho E + p)) = 0, \end{cases}$$

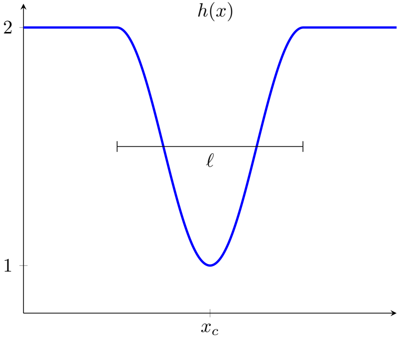
+b.c.

Cost functional: $J(\mathbf{U}) = \frac{1}{2}\|p - p^*\|^2_{L^2}$

Parameters: $\mathbf{a} = (x_c, \ell)^t$

Target pressure: $p^* = p(x_c = 0.5, \ell = 0.5)$

Gradient: $\nabla_{\mathbf{a}} J(\mathbf{U}) = \begin{bmatrix} (p - p^*, p_{x_c})_{L^2} \\ (p - p^*, p_\ell)_{L^2} \end{bmatrix}$



Optimisation problem: $\min_{\mathbf{a} \in \mathcal{A}} J(\mathbf{U})$ subject to (1).

Optimisation

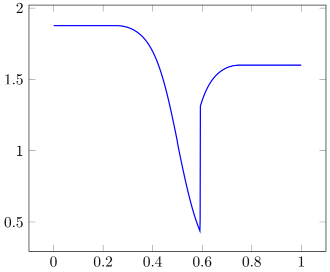
Boundary conditions:

- inlet: enthalpy H_L and total pressure $p_{tot,L}$
- outlet: pressure p_R

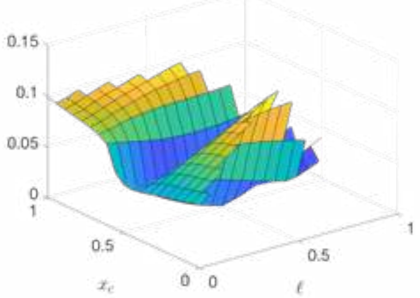
$$\begin{cases} H &= E + \dfrac{p}{\rho}, \\ p &= (\gamma - 1)\left(\rho E - \dfrac{1}{2}\rho u^2\right), \\ p_{tot} &= p + \dfrac{1}{2}\rho u^2. \end{cases}$$

These b.c. provide a discontinuous solution [6] $\forall \mathbf{a} \in \mathcal{A}$.

$p^*(x)$

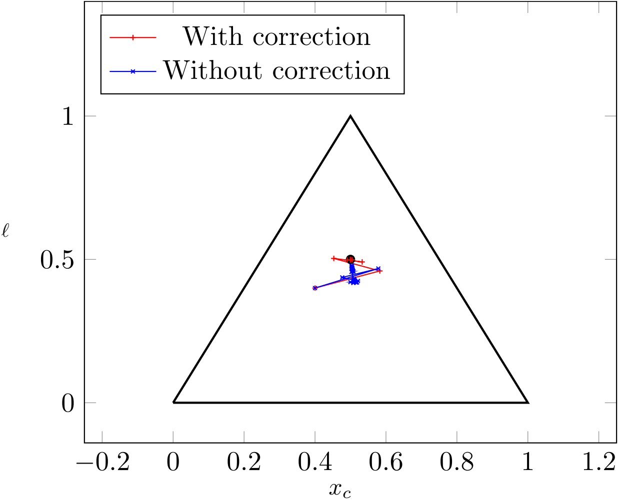


$J(x_c, \ell)$

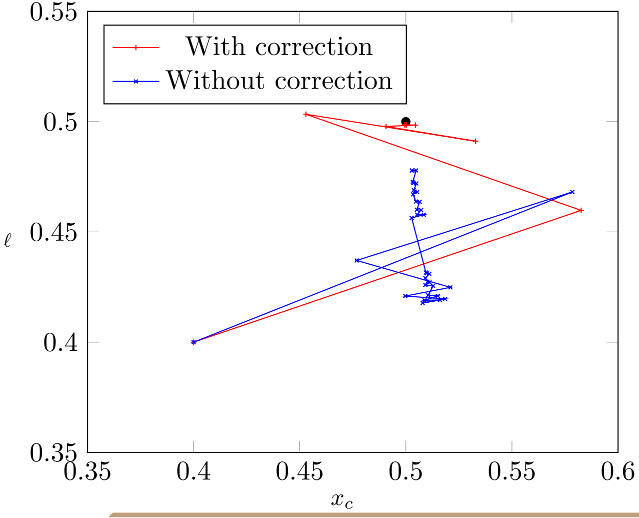


[6] Giles, M. B., & Pierce, N. A. (2001). Analytic adjoint solutions for the quasi-one-dimensional Euler equations. *Journal of Fluid Mechanics*, *426*, 327-345.

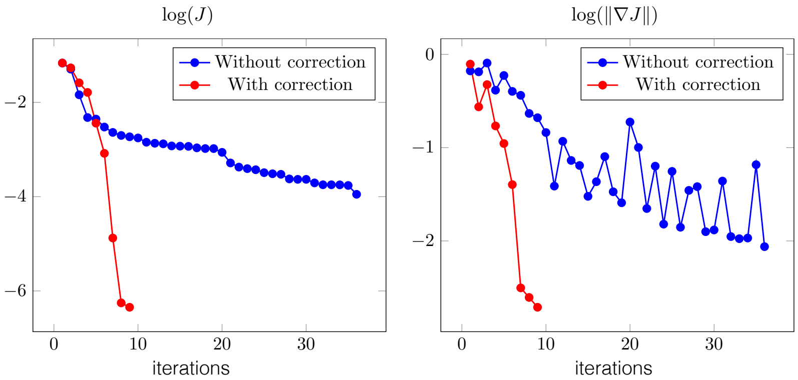
Optimisation



Optimisation



Optimisation



Uncertainty propagation for the Navier-Stokes equations

Camilla Fiorini [1]

In collaboration with

Maria Adela Puscas [2]

Bruno Després [3]

[1] INRIA, Rennes [2] CEA Saclay [3] LJLL, Sorbonne Université

SA for Navier-Stokes

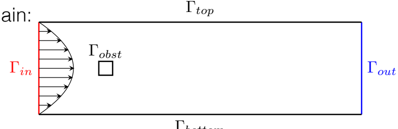
State equations:

$$\begin{aligned}
&\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, && \Omega, t>0\\
&\nabla\cdot\mathbf{u} = 0, && \Omega, t>0\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{d}(x), && \Omega, t=0\\
&\mathbf{u} = -g(y)\mathbf{n}, && \text{on } \Gamma_{in}\\
&\mathbf{u} = 0, && \text{on } \Gamma_w\\
&(\nu\nabla\mathbf{u} - pI)\mathbf{n} = 0, && \text{on } \Gamma_{out}
\end{aligned}$$

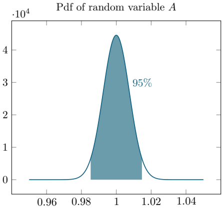
Sensitivity equations

$$\begin{aligned}
\partial_t \mathbf{u}_a - \nu\Delta\mathbf{u}_a + (\mathbf{u}_a\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_a + \nabla p_a &= \mathbf{f}_a,\\
\nabla\cdot\mathbf{u}_a &= 0,\\
\mathbf{u}_a(\mathbf{x},0) &= \mathbf{d}_a(x),\\
\mathbf{u}_a &= -g_a(y)\mathbf{n},\\
\mathbf{u}_a &= 0,\\
(\nu\nabla\mathbf{u}_a - p_a I)\mathbf{n} &= 0.
\end{aligned}$$

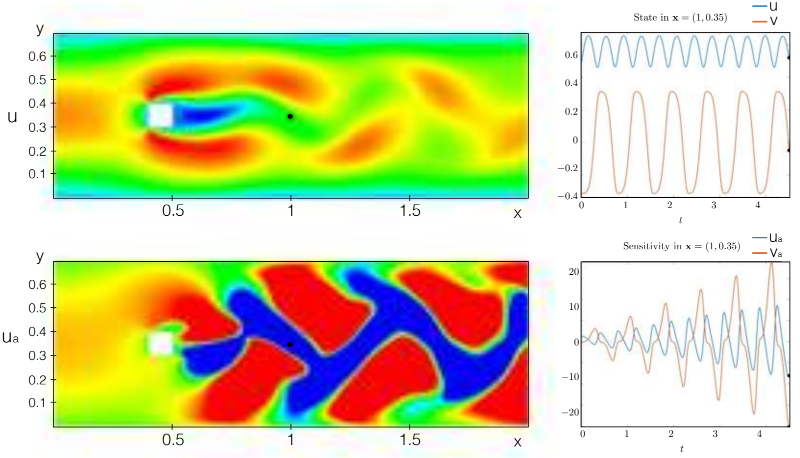
- Inlet velocity $g(y) = \frac{4A}{\ell^2} y(\ell - y)$ Uncertain parameter : amplitude A.
- Numerical method : finite elements volumes
- Unstructured 2D mesh
- Domain:



$\Gamma_w := \Gamma_{obst} \cup \Gamma_{top} \cup \Gamma_{bottom}$

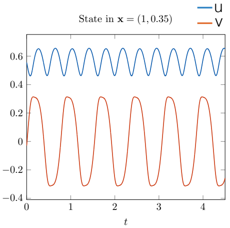


Unsteady test case



Unsteady test case

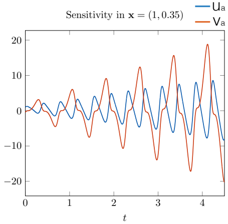
$$\mathbf{u}(\mathbf{x},t;a) \simeq \sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x};a)\cos(\omega_k(a)t).$$



amplitude sensitivity | **frequency sensitivity**

$$\mathbf{u}_a(\mathbf{x},t;a) \simeq \sum_{k=0}^{N} \boxed{\mathbf{u}_{0,a,k}(\mathbf{x};a)}\cos(\omega_k(a)t) - t\sum_{k=0}^{N} \mathbf{u}_{0,k}(\mathbf{x};a)\boxed{\omega'_k(a)}\sin(\omega_k(a)t)$$

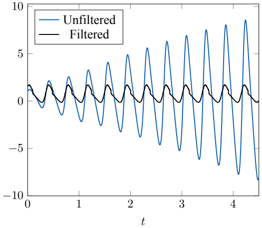
Bounded | **Unbounded**



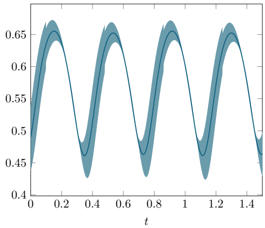
[3] C. Fiorini, B. Després, M. A. Puscas. Sensitivity equation method for the Navier–Stokes equations applied to uncertainty propagation. *International Journal for Numerical Methods in Fluids*, 93(1):71-92, 2021.
[4] H. Hristova, S. Etienne, D. Pelletier, J. Borggaard. A continuous sensitivity equation method for time-dependent incompressible laminar flows. *International Journal for Numerical Methods in Fluids*, 50:817–844, 2004.

Unsteady test case

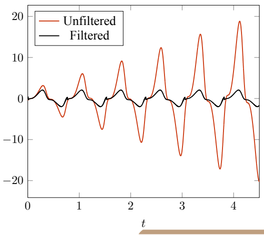
u_a in $\mathbf{x} = (1, 0.35)$



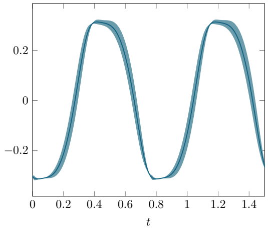
Confidence interval for u in $\mathbf{x} = (1, 0.35)$



v_a in $\mathbf{x} = (1, 0.35)$



Confidence interval for v in $\mathbf{x} = (1, 0.35)$



Milstein scheme for SQG under LU

Camilla Fiorini [1]

In collaboration with

Long Li [1]

Etienne Mémin [1]

[1] Fluminance, INRIA, Rennes

SQG system under location uncertainty

LU framework: based on the following decomposition of the Lagrangian velocity in two components

$$\mathrm{d}\mathbf{X}_t = \mathbf{u}(\mathbf{X}_t, t)\mathrm{d}t + \sigma(\mathbf{X}_t, t)\mathrm{d}\mathbf{B}_t$$

one can compute the **stochastic transport operator**:

$$\mathbb{D}_t b := \mathrm{d}_t b + \mathbf{v}^* \cdot \nabla b \ \mathrm{d}t + \sigma \mathrm{d}\mathbf{B}_t \cdot \nabla b - \frac{1}{2}\nabla \cdot (a \nabla b)\mathrm{d}t,$$

where

$$\mathbf{v}^* = \mathbf{u} - \frac{1}{2}\nabla \cdot a - \sigma(\nabla \cdot \sigma)$$

Therefore, the surface quasi geostrophic system under location uncertainty is:

$$\begin{cases} \mathbb{D}_t b = 0, \\ b = N(-\Delta)^{1/2}\psi, \\ \mathbf{u} = \nabla^{\perp}\psi, \end{cases}$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \; \mathrm{d}s - \int_{t_0}^{t} \nabla b \cdot \sigma \mathrm{d}\mathbf{B}_s,$$

$$\sigma \mathrm{d}\mathbf{B}_s \rightarrow \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We define the following functions:

$$f(b_t, t) = \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \qquad g^m(b_t, t) = \nabla b \cdot \varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t, t) = f(b_{t_0}, t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial f}{\partial s}}_{=0}(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

$$g^m(b_t, t) = g^m(b_{t_0}, t_0) + \int_{t_0}^{t} \underbrace{\frac{\partial g}{\partial s}^m}_{=0}(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g}{\partial b}^m(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g}{\partial b^2}^m(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

Towards the Milstein scheme

The main equation is:

$$b_t = b_{t_0} + \int_{t_0}^{t} \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \,\mathrm{d}s - \int_{t_0}^{t} \nabla b\cdot \sigma \mathrm{d}\mathbf{B}_s,$$

$$\sigma \mathrm{d}\mathbf{B}_s \rightarrow \sum_m \varphi_m \mathrm{d}\beta_s^m,$$

We define the following functions:

$$f(b_t, t) = \frac{1}{2}\nabla\cdot(a\nabla b) - \mathbf{v}^*\cdot\nabla b \qquad g^m(b_t, t) = \nabla b\cdot\varphi_m$$

We can apply Itō formula for f and g^m, obtaining:

$$f(b_t, t) = f(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial s}(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial f}{\partial b}(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 f}{\partial b^2}(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

$$g^m(b_t, t) = g^m(b_{t_0}, t_0) + \int_{t_0}^{t} \frac{\partial g}{\partial s}^m(b_s, s)\mathrm{d}s + \int_{t_0}^{t} \frac{\partial g}{\partial b}^m(b_s, s)\mathrm{d}b_s + \frac{1}{2}\int_{t_0}^{t} \frac{\partial^2 g}{\partial b^2}^m(b_s, s)\mathrm{d}\langle b, b\rangle_s$$

Milstein scheme

By replacing everything in the Itō formulas and then into the main equation, one finds:

$$b_t = b_{t_0} + f(b_{t_0})\Delta t - \sum_m g^m(b_{t_0})\Delta\beta^m + \int_{t_0}^{t}\int_{t_0}^{s}\sum_{m,k} g^m(g^k(b_\tau))\mathrm{d}\beta_\tau^k \mathrm{d}\beta_s^m \quad (1)$$

Euler-Maruyama

We define the following quantities:

$$G^{m,k} := g^m(g^k(b_{t_0})) \qquad I^{m,k} := \int_{t_0}^{t}\int_{t_0}^{s} \mathrm{d}\beta_\tau^k \mathrm{d}\beta_s^m$$

Then the double integral in (1) can be approximated with:

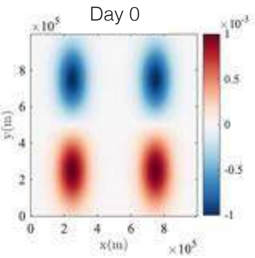
$$\sum_{m,k} G^{m,k} I^{m,k} = \sum_{m,k} G^{m,k}\underbrace{\frac{I^{m,k}+I^{k,m}}{2}}_{= \Delta\beta^m\Delta\beta^k - \delta_{m,k}\Delta t} + G^{m,k}\frac{I^{m,k}-I^{k,m}}{2}$$

Lévy area, which can be simulated → weak approximation / recursive (conditional) approximation / neglected

Remark: if G is symmetric (i.e. $G^{m,k} = G^{k,m}$), then the Lévy area is not necessary:

$$\sum_{m,k} G^{m,k} I^{m,k} = \frac{1}{2}\sum_{m,k} G^{m,k}I^{m,k} + G^{k,m}I^{k,m} = \sum_{m,k} G^{m,k}\frac{I^{m,k}+I^{k,m}}{2}$$

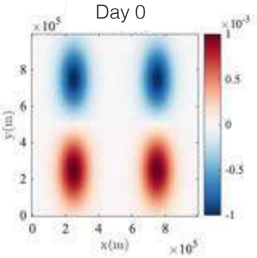
Deterministic high resolution



Spatial resolution : 512x512

Time scheme RK4

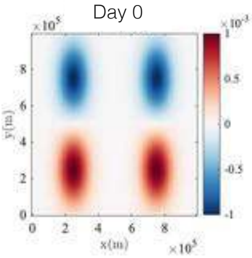
Euler Maruyama



Spatial resolution: 128x128

Time scheme order 0.5

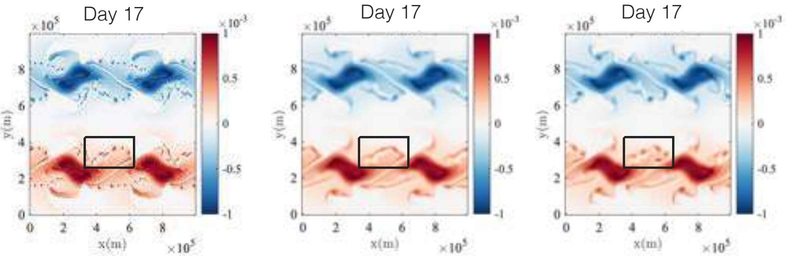
Milstein



Spatial resolution: 128x128

Time scheme order 1

Deterministic high resolution | **Euler Maruyama** | **Milstein**



Deterministic high resolution	Euler Maruyama	Milstein
Spatial resolution : 512x512	Spatial resolution: 128x128	Spatial resolution: 128x128
Time scheme RK4	Time scheme order 0.5	Time scheme order 1

Thank you
for your attention