Sensitivity analysis for the Euler equations in Lagrangian coordinates

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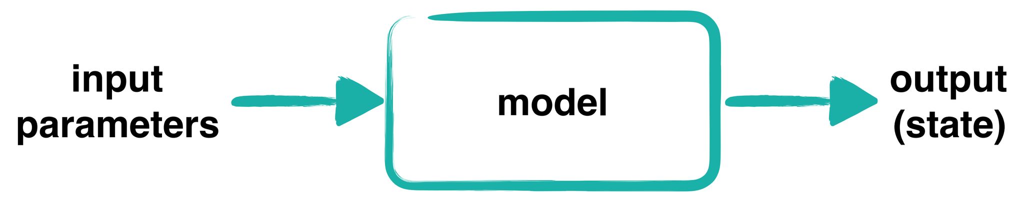
Introduction to Sensitivity Analysis

Sensitivity Analysis: study of how changes in the **inputs** of a model affect the **outputs**.



· Parameter of interest: a,

· State: $\mathbf{U}$,

· Model: system of hyperbolic equations.

Sensitivity: $\dfrac{\partial \mathbf{U}}{\partial a} = \mathbf{U}_a$

Applications

- Optimization: $\min_{a\in\mathcal{A}} J(\mathbf{U})$, where $J(\mathbf{U}) = \frac{1}{2}b(\mathbf{U},\mathbf{U})$ and b bilinear.
  Classical optimization techniques call for the differentiation of the cost function:
  $$\frac{\partial J(\mathbf{U})}{\partial a} = b(\mathbf{U},\mathbf{U}_a).$$
- Quick evaluation of close solutions [3]:
  $$\mathbf{U}(a+\delta a) = \mathbf{U}(a) + \delta a \mathbf{U}_a(a) + o(\delta a^2).$$
- Uncertainty quantification [3]:
  $$\mu \quad \mathbf{U}(\mu_a),$$
  $$\sigma^2 \quad \mathbf{U}_a(\mu_a)^T \mathbf{U}_a(\mu_a)\sigma_a^2.$$

Continuous Sensitivity Equation Method

Standard technique under hypothesis of regularity of U

$$\begin{cases} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0} & \Omega\times(0,T) \\ \mathbf{U}(x,0) = \mathbf{U}_0(x) & \Omega \end{cases} \quad -\partial_a \rightarrow \quad \begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U},\mathbf{U}_a) = \mathbf{0} & \Omega\times(0,T) \\ \mathbf{U}_a(x,0) = \mathbf{U}_{a,0}(x) & \Omega \end{cases}$$

If these techniques are applied to hyperbolic equations in case of shocks, **Dirac delta functions** appear in the sensitivity. In order to have a sensitivity system which is valid also when the state is discontinuous, we add a **correction term** [4].

Sensitivity system with correction term

$$\begin{cases} \partial_t \mathbf{U}_a + \partial_x \mathbf{F}_a(\mathbf{U},\mathbf{U}_a) = \mathbf{S} & \Omega\times(0,T) \\ \mathbf{U}_a(x,0) = \mathbf{U}_{a,0}(x) & \Omega \end{cases} \qquad (1)$$

This system is valid in the usual sense of weak solutions also in the case of discontinuous state variables [1].

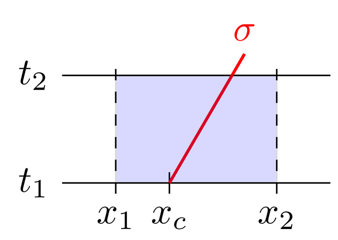
Source term definition

$$\mathbf{S} = \sum_{i=1}^{N_s} \boldsymbol{\rho}_k(t)\delta(x - x_{s,k}(t))$$

· N_s: number of discontinuities,

· $x_{k,s}(t)$: position of the k^{th} shock at time t,

· $\boldsymbol{\rho}_k$: amplitude of the k^{th} correction.



Integrating the sensitivity equations over the control volume: $\boldsymbol{\rho} = (\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma + \mathbf{F}_a^+ - \mathbf{F}_a^-$.

Rankine-Hugoniot conditions for the state: $(\mathbf{U}^+ - \mathbf{U}^-)\sigma = \mathbf{F}^+ - \mathbf{F}^-$.

Differentiating them with respect to the parameter one has:

$$(\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + (\mathbf{U}^- - \mathbf{U}^+)\sigma_{k,a} + \sigma_k(\nabla\mathbf{U}^+ - \nabla\mathbf{U}^-)\partial_a x_{k,s}(t) =$$
$$= \mathbf{F}_a^- - \mathbf{F}_a^+ + \left(\frac{\partial \mathbf{F}(\mathbf{U}^+)}{\partial \mathbf{U}}\nabla\mathbf{U}^+ - \frac{\partial \mathbf{F}(\mathbf{U}^-)}{\partial \mathbf{U}}\nabla\mathbf{U}^-\right)\partial_a x_{k,s}(t).$$

The terms in green are difficult to estimate due to $\partial_a x_{k,s}(t)$.

They are zero if we consider a solution $\mathbf{U}$ which is **constant** in the left and right neighbourhoods of the shock. We obtain therefore a simpler formula: $(\mathbf{U}_a^- - \mathbf{U}_a^+)\sigma_k + (\mathbf{U}^- - \mathbf{U}^+)\sigma_{k,a} = \mathbf{F}_a^- - \mathbf{F}_a^+$.

Finally, we have:

$$\boldsymbol{\rho}_k(t) = \sigma_{k,a}(\mathbf{U}^+ - \mathbf{U}^-).$$

Remark: shock detectors are necessary.

Bibliography

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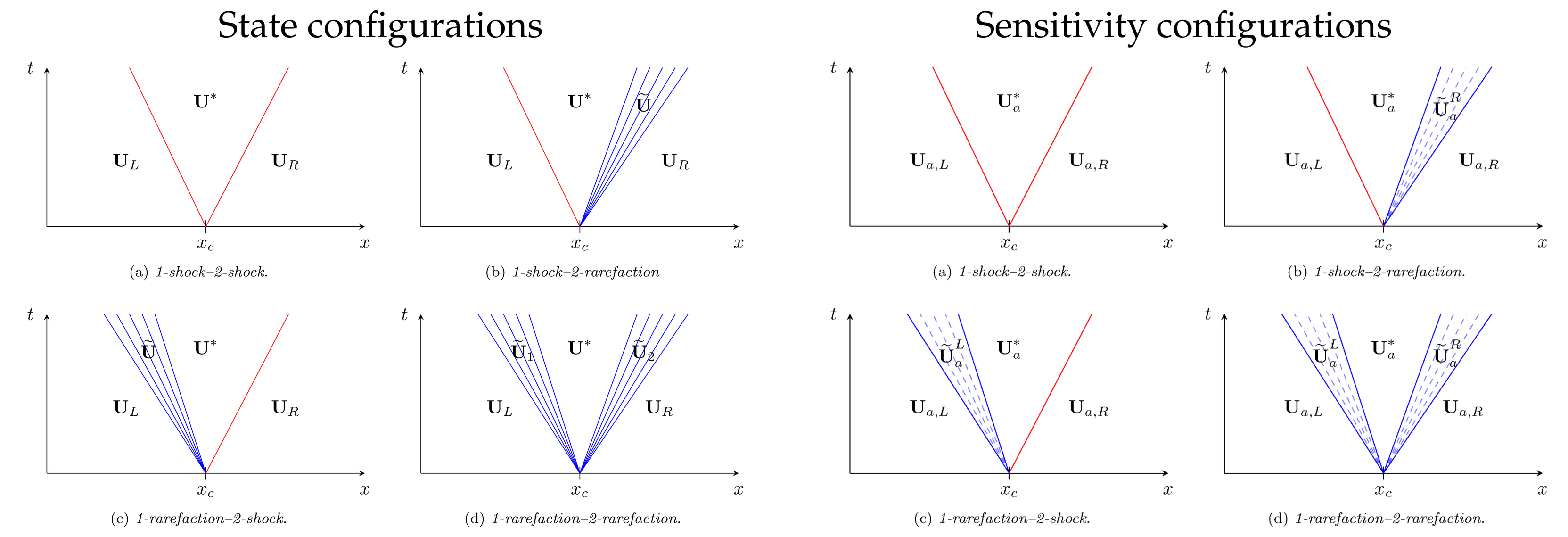
Solution of the Riemann problem

The $p-$system and its sensitivity

$$\begin{cases} \partial_t \tau - \partial_x u = 0 & \Omega\times(0,T), \\ \partial_t u + \partial_x p(\tau) = 0 & \Omega\times(0,T), \\ \tau(x,0) = \tau_L \mathbb{1}_{x<x_c} + \tau_R \mathbb{1}_{x>x_c} & \Omega, \\ u(x,0) = u_L \mathbb{1}_{x<x_c} + u_R \mathbb{1}_{x>x_c} & \Omega. \end{cases} \qquad \begin{cases} \partial_t \tau_a - \partial_x u_a = 0 & \Omega\times(0,T), \\ \partial_t u_a + \partial_x (p'(\tau)\tau_a) = 0 & \Omega\times(0,T), \\ \tau(x,0) = \tau_{a,L} \mathbb{1}_{x<x_c} + \tau_{a,R} \mathbb{1}_{x>x_c} & \Omega, \\ u(x,0) = u_{a,L} \mathbb{1}_{x<x_c} + u_{a,R} \mathbb{1}_{x>x_c} & \Omega. \end{cases}$$

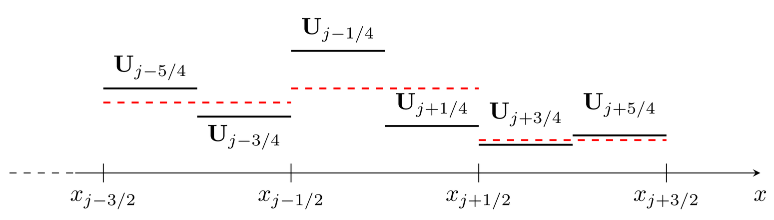
State configurations | Sensitivity configurations



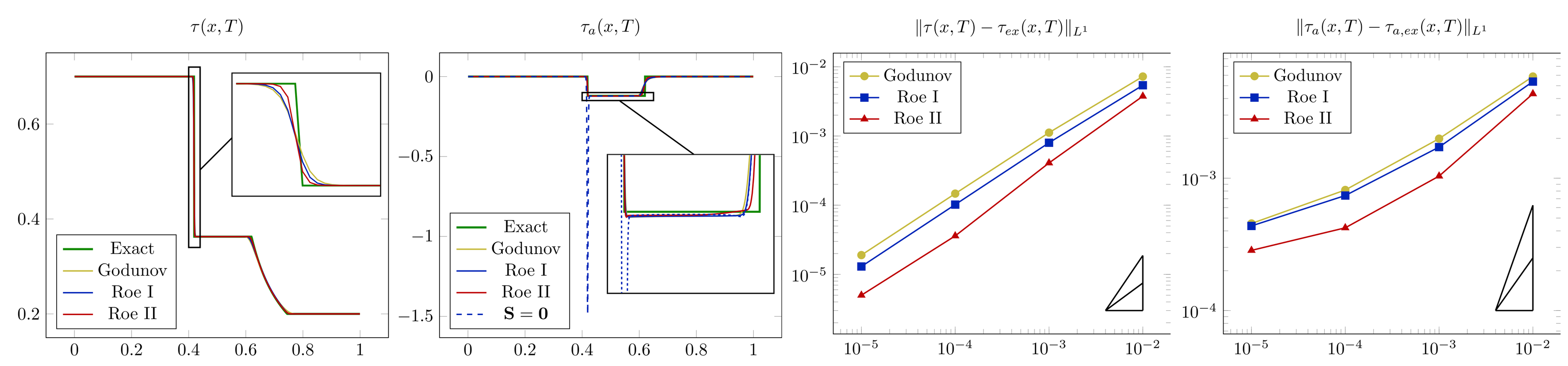
Classical numerical methods

- Exact Godunov-type scheme
  State: $\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \dfrac{\Delta t}{\Delta x}(\mathbf{F}(\mathbf{U}^*_{j+1/2}) - \mathbf{F}(\mathbf{U}^*_{j-1/2}))$. Sensitivity: direct average.
- First order Roe-type scheme
  State: $\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \dfrac{\Delta t}{\Delta x}(\lambda^{ROE}_{j-1/2}(\mathbf{U}^*_{j-1/2} - \mathbf{U}_j^n) + \lambda^{ROE}_{j+1/2}(\mathbf{U}^*_{j+1/2} - \mathbf{U}_j^n))$.
  Sensitivity: $\mathbf{U}^*_{a,j-1/2} = \dfrac{1}{2}(\mathbf{U}^n_{a,j-1} + \mathbf{U}^n_{a,j}) - \dfrac{\mathbf{F}_a(\mathbf{U}_j^n,\mathbf{U}^n_{a,j}) - \mathbf{F}_a(\mathbf{U}^n_{j-1},\mathbf{U}^n_{a,j})}{2\lambda^{ROE}_{j-1/2}} + \dfrac{\Delta x \mathbf{S}^n_{j-1/2}}{2\lambda^{ROE}_{j-1/2}}$.
- Second order Roe-type scheme
  Time discretisation: two-step Runge Kutta. Space discretisation: MUSCL-type scheme.

$$\mathbf{U}^n_{j\pm1/4} = \mathbf{U}_j \pm \Delta\mathbf{U}^n_j,$$
$$\Delta\mathbf{U}^n = \tfrac{1}{2}\text{minmod}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j, \mathbf{U}^n_j - \mathbf{U}^n_{j-1}),$$
$$\text{minmod}(a,b) = \text{sgn}(a)\min(|a|,|b|)\mathbb{1}_{ab>0}$$



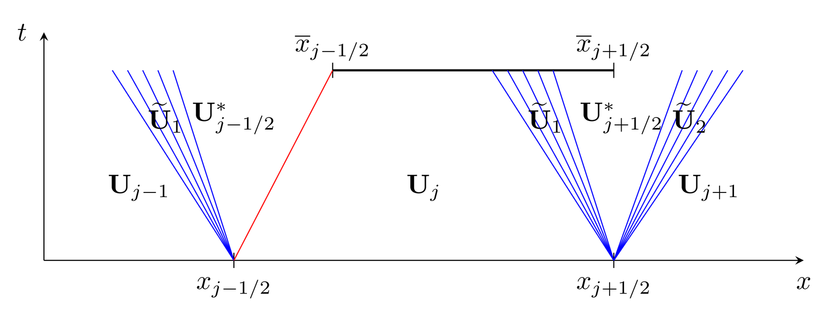
Numerical results



Anti-diffusive numerical method

A modified Godunov-type method [2]:

1. Initial data discretisation
2. Solution of a Riemann problem for each interface
3. Definition of a staggered mesh on which the average is performed



$$\overline{x}^n_{j-1/2} = x_{j-1/2} + \sigma^n_{j-1/2}\Delta t^n,$$

$$\sigma^n_{j+1/2} = \begin{cases} \lambda^n_{j+1/2} & u_j > u_{j+1} \text{ and } \tau_j < \tau_{j+1}, \\ -\lambda^n_{j+1/2} & u_j > u_{j+1} \text{ and } \tau_j > \tau_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

4. Projection onto the initial mesh:

$$\mathbf{U}_j = \begin{cases} \overline{\mathbf{U}}_{j-1} & \text{if } \alpha \in \left(0, \frac{\Delta t}{\Delta x}\max(\sigma_{j-1/2},0)\right), \\ \overline{\mathbf{U}}_j & \text{if } \alpha \in \left[\frac{\Delta t}{\Delta x}\max(\sigma_{j-1/2},0), 1 + \frac{\Delta t}{\Delta x}\min(\sigma_{j+1/2},0)\right), \\ \overline{\mathbf{U}}_{j+1} & \text{if } \alpha \in \left[1 + \frac{\Delta t}{\Delta x}\min(\sigma_{j+1/2},0), 1\right). \end{cases} \qquad \alpha \sim \mathcal{U}([0,1])$$

Numerical results