Uncertainty quantification for the Navier–Stokes equations

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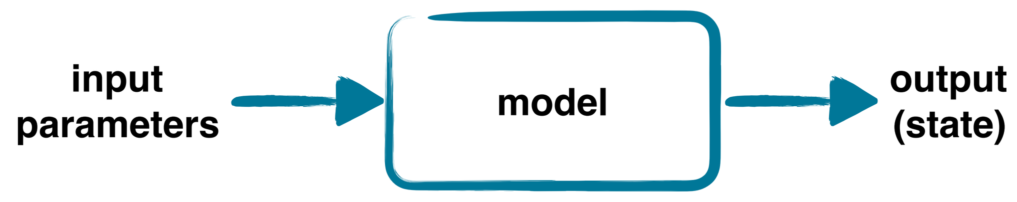
Introduction to Sensitivity Analysis

Sensitivity Analysis: study of how changes in the **inputs** of a model affect the **outputs**.



· Parameter of interest: a,
· State: $\mathbf{u}$, p,
· Model: Navier–Stokes system.

Sensitivities: $\mathbf{u}_a := \dfrac{\partial \mathbf{u}}{\partial a}$, $p_a := \dfrac{\partial p}{\partial a}$.

Continuous Sensitivity Equation Method

State equations

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \Omega, t>0, \\ \nabla\cdot\mathbf{u} = 0 & \Omega, t>0, \\ \mathbf{u}(\mathbf{x},0) = 0 & \Omega, t=0, \\ \mathbf{u} = -g(y)\mathbf{n} & \text{on } \Gamma_{in}, \\ \mathbf{u} = 0 & \text{on } \Gamma_w, \\ (\nu\nabla\mathbf{u} - pI)\mathbf{n} = 0 & \text{on } \Gamma_{out}. \end{cases}$$

The continuous sensitivity equation method [1] is a *differentiate-then-discretise* technique, which consists in formally differentiating the state system with respect to the parameter of interest a and then exchanging the derivatives in space and time with the ones in a. For the Navier–Stokes equations, one obtains the following system:

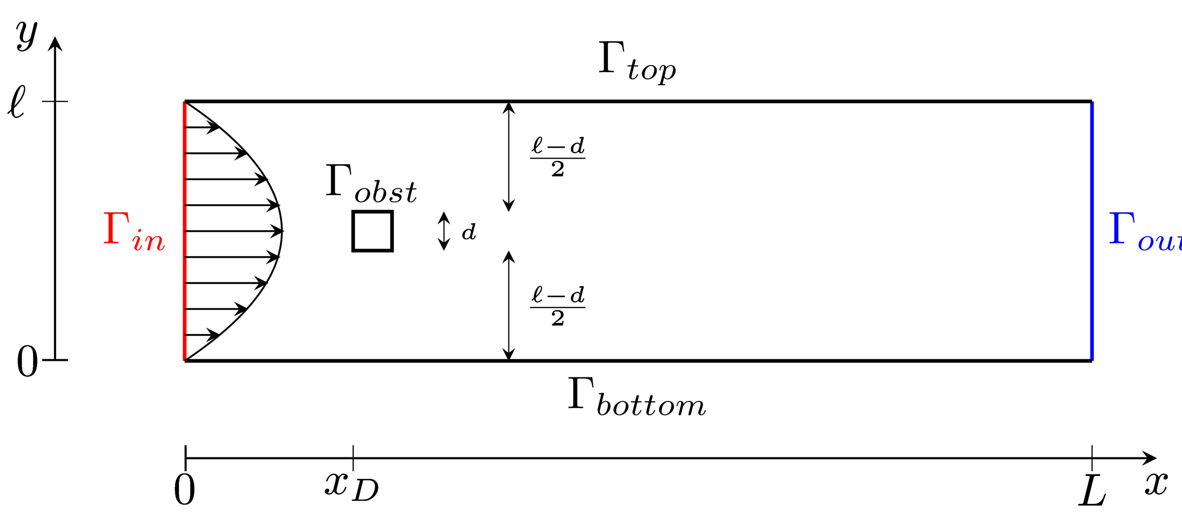
Sensitivity equations

$$\begin{cases} \partial_t \mathbf{u}_a - \nu \Delta \mathbf{u}_a + (\mathbf{u}_a\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_a + \nabla p_a = \bar{\mathbf{f}}_a & \Omega, t>0, \\ \nabla\cdot\mathbf{u}_a = 0 & \Omega, t>0, \\ \mathbf{u}_a(\mathbf{x},0) = 0 & \Omega, t=0, \\ \mathbf{u}_a = -g_a(y)\mathbf{n} & \text{on } \Gamma_{in}, \\ \mathbf{u}_a = 0 & \text{on } \Gamma_w, \\ (\nu\nabla\mathbf{u}_a - p_a I)\mathbf{n} = -\nu_a\nabla\mathbf{u}\,\mathbf{n} & \text{on } \Gamma_{out}, \end{cases}$$

where $\bar{\mathbf{f}}_a := \partial_a \mathbf{f} + \nu_a \Delta \mathbf{u}$.

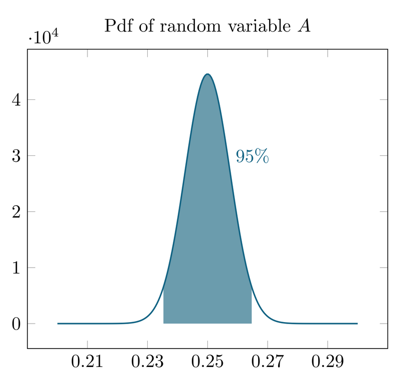
Test case

We consider the domain Ω here below:



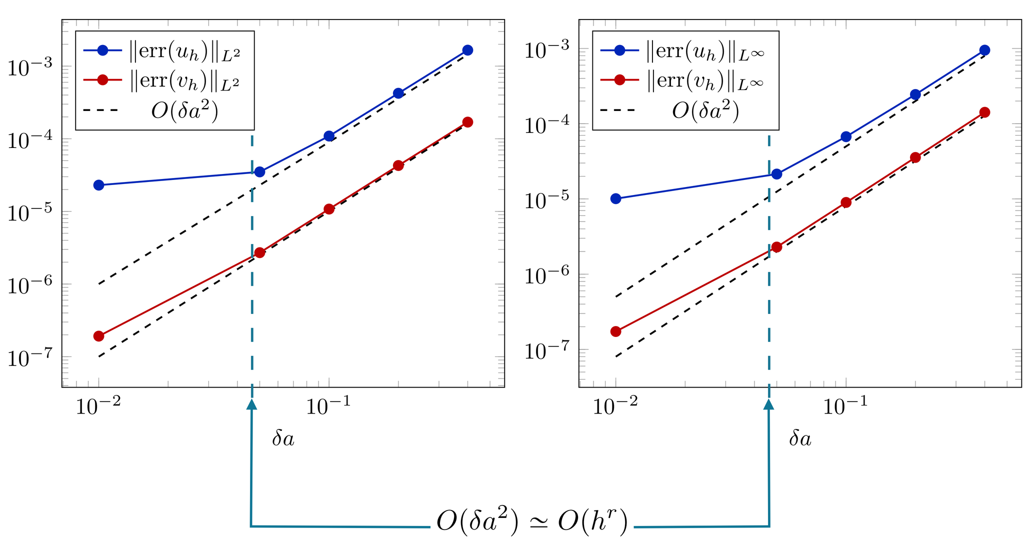
with $\Gamma_w := \Gamma_{top} \cup \Gamma_{bottom} \cup \Gamma_{obst}$.

The inlet velocity is $g(y) = \dfrac{4A}{\ell^2} y(\ell - y)$, where A is the uncertain parameter (i.e. $a = A$ in this test case). A is a gaussian random variable of mean μ_A and variance σ_A^2. Its probability density function is plotted on the right.



Validation

From a first order Taylor expansion, one can define the following quantity:
$\mathrm{err}(\mathbf{u}) = \mathbf{u}(x,T;a+\delta a) - \mathbf{u}(x,T;a) - \delta a \mathbf{u}_a(x,T;a) \simeq O(\delta a^2)$.

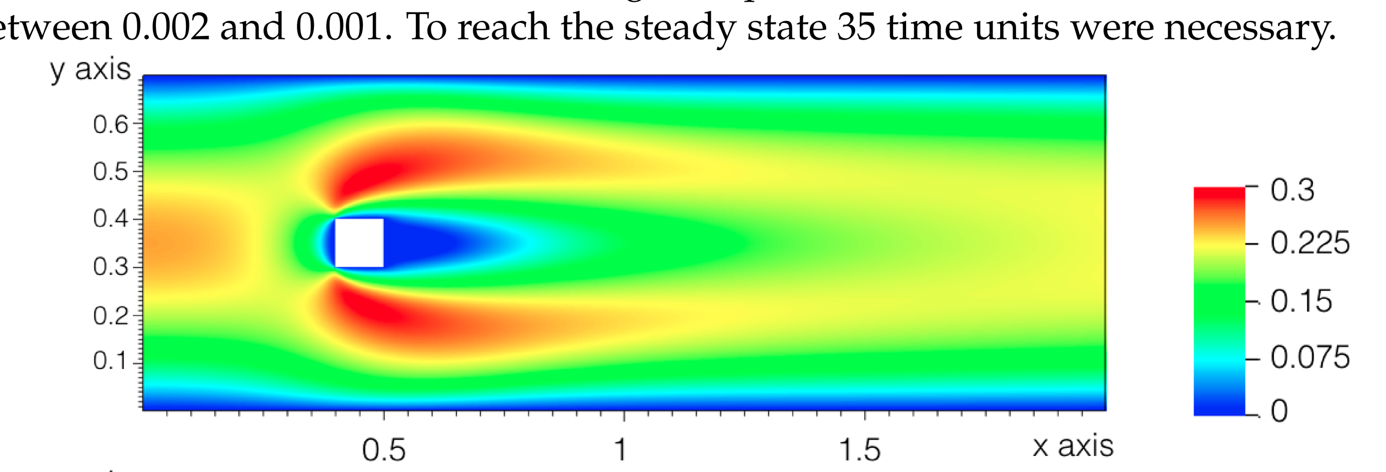


In these Figures, we show the L^2 and L^∞ norms of the error for each component of the velocity. The change in slope occurs when the error due to the spatial discretisation becomes comparable to the one due to the Taylor expansion here above.

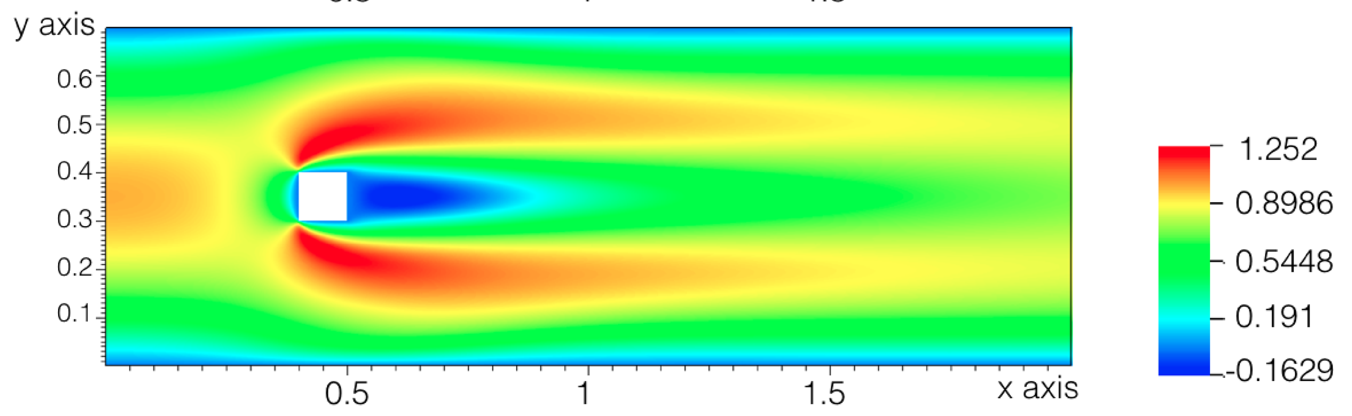
Numerical results

The following results are obtained with TrioCFD, using an explicit Euler scheme in time, on a mesh with h varying between 0.002 and 0.001. To reach the steady state 35 time units were necessary.

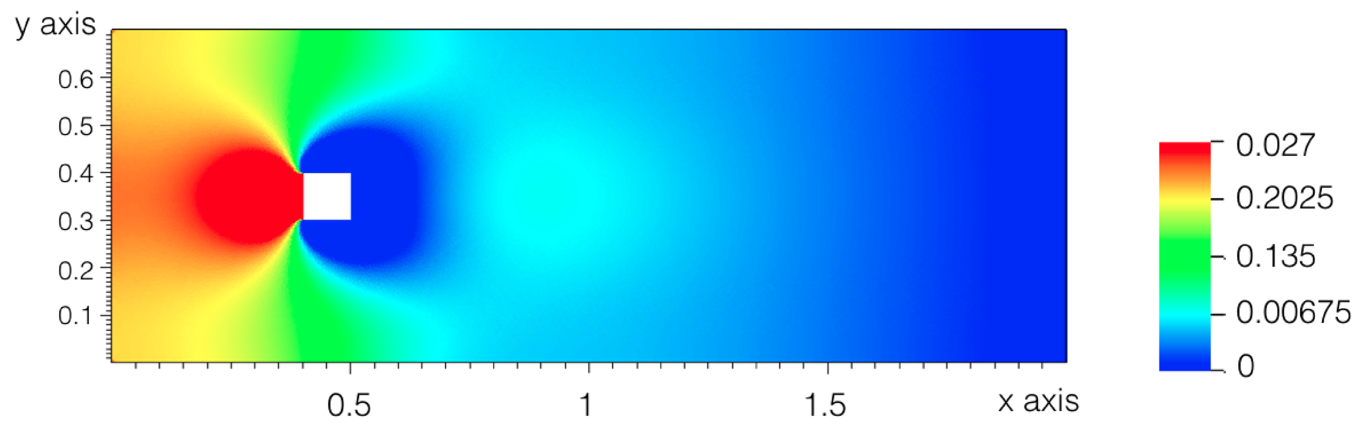
$x-$component of the velocity



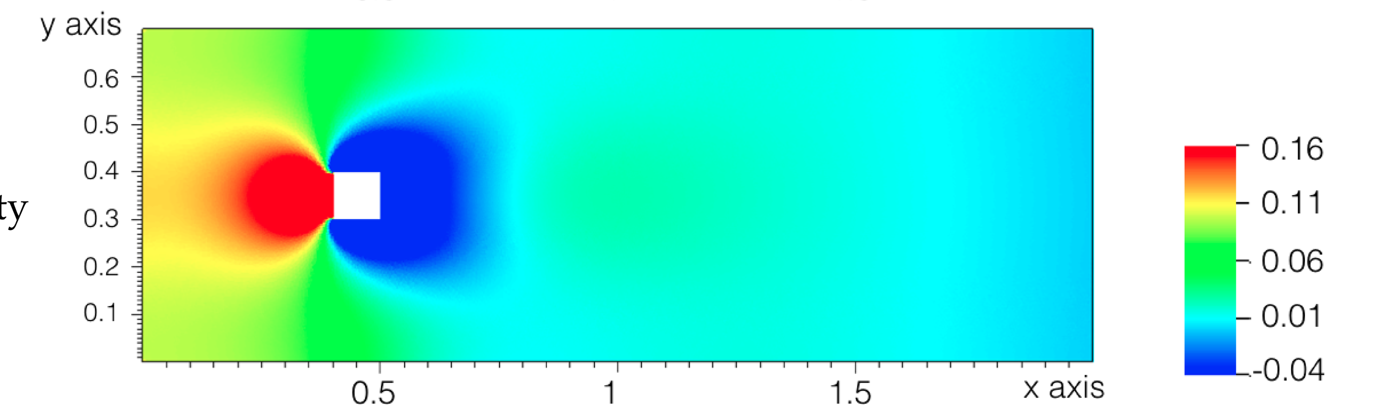
$x-$component of the velocity sensitivity



pressure



pressure sensitivity



Uncertainty quantification

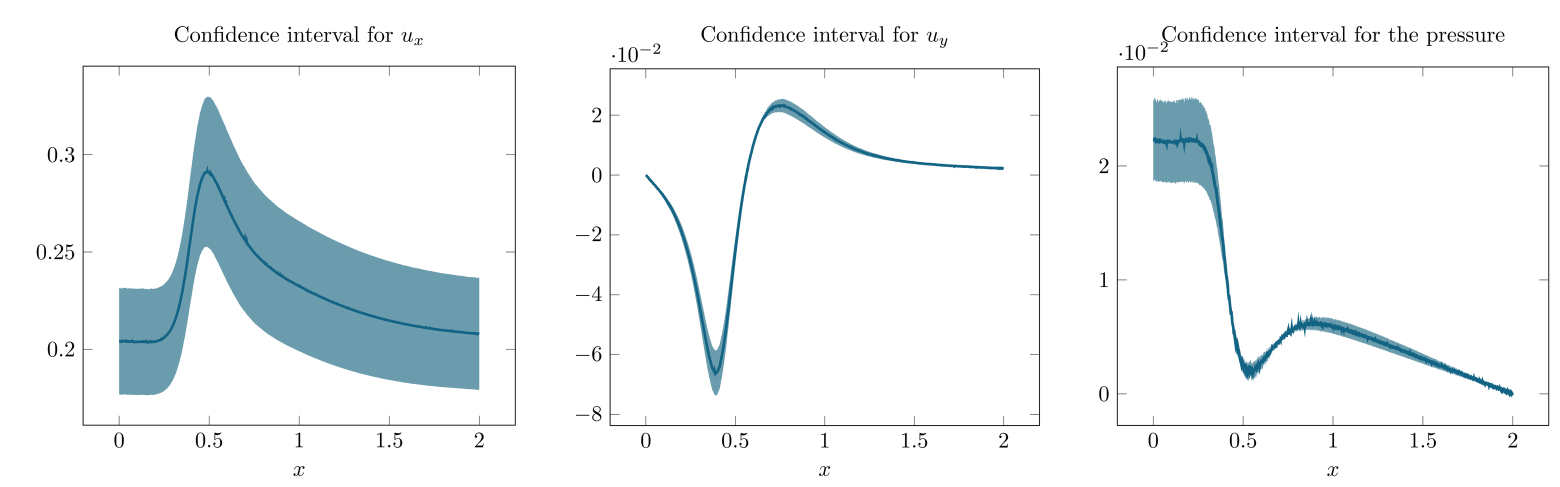
The aim is to determine a **confidence interval** CI_X for a variable X, such that $P(X \in CI_X) \geq 1 - \alpha$. From the Chebyshev's inequality we have $CI_X = \left[\mu_X - \dfrac{\sigma_X}{\sqrt{\alpha}}, \mu_X + \dfrac{\sigma_X}{\sqrt{\alpha}}\right]$.

SA provides us with the following **first order estimates** of the mean μ_X and the variance σ_X^2 [2]

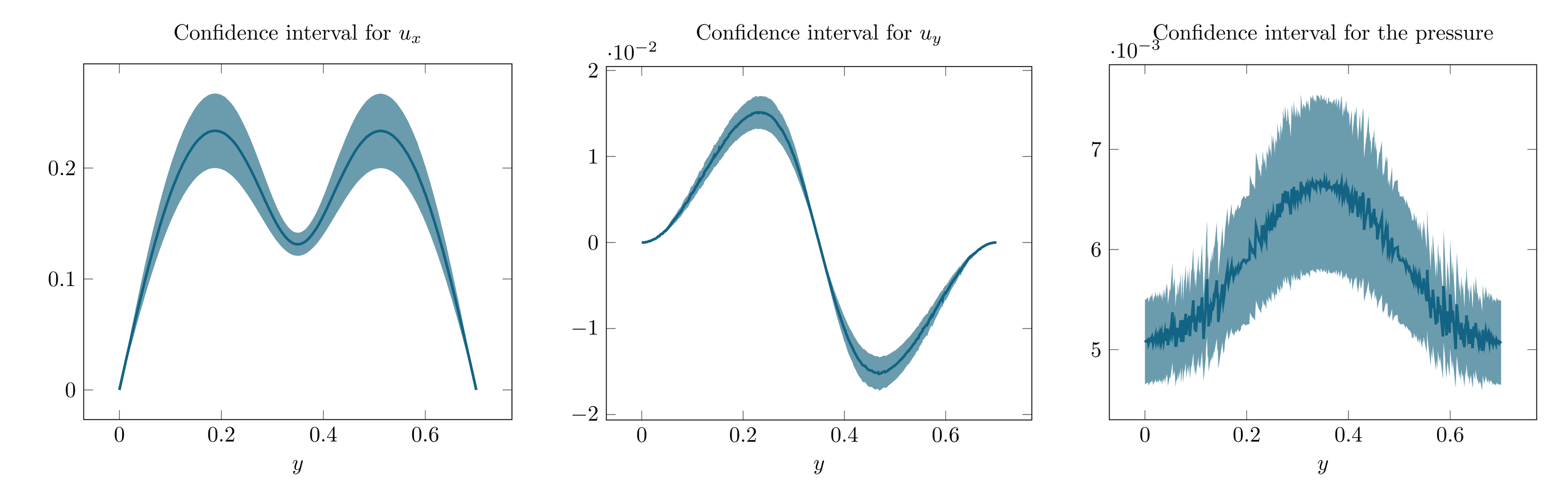
$$\mu_X = X(\mu_a), \quad \sigma_X^2 = X_a^2(\mu_a)\sigma_a^2,$$

which require only one simulation of the state and one of the sensitivity.

CI on the horizontal cross section $y = 0.2$ for velocity and pressure



CI on the vertical cross section $x = 1$ for velocity and pressure



Bibliography

[1] Régis Duvigneau, Dominique Pelletier, and Jeff Borggaard. An improved continuous sensitivity equation method for optimal shape design in mixed convection. *Numerical Heat Transfer, Part B: Fundamentals*, 50(1):1–24, 2006.

[2] Camilla Fiorini. *Sensitivity analysis for nonlinear hyperbolic systems of conservation laws*. PhD thesis, Université de Versailles Saint-Quentin-En-Yvelines, 2018.